4 EXPECTATIONS AND MOMENTS

An extremely useful concept in problems involving random variables or distributions is that of *expectation*. The subsections of this section give definitions and results regarding expectations.

4.1 Mean

Definition 10 Mean Let X be a random variable. The *mean* of X, denoted by μ_X or $\mathscr{E}[X]$, is defined by:

(i) $$\mathscr{E}[X] = \sum x_j f_X(x_j) \tag{4}$$

if X is discrete with mass points $x_1, x_2, \ldots, x_j, \ldots$.

(ii) $$\mathscr{E}[X] = \int_{-\infty}^{\infty} x f_X(x) dx \tag{5}$$

if X is continuous with probability density function $f_X(x)$.

(iii) $$\mathscr{E}[X] = \int_0^{\infty} [1 - F_X(x)]\, dx - \int_{-\infty}^{0} F_X(x)\, dx \tag{6}$$

for an arbitrary random variable X. ////

EXAMPLE 10 Consider the experiment of tossing two dice. Let X denote the total of the two dice and Y their absolute difference. The discrete density functions for X and Y are given in Example 6.

$$\mathscr{E}[Y] = \sum y_j f_Y(y_j) = \sum_{i=0}^{5} i f_Y(i) = 0 \cdot \tfrac{6}{36} + 1 \cdot \tfrac{10}{36}$$
$$+ 2 \cdot \tfrac{8}{36} + 3 \cdot \tfrac{6}{36} + 4 \cdot \tfrac{4}{36} + 5 \cdot \tfrac{2}{36} = \tfrac{70}{36}.$$
$$\mathscr{E}[X] = \sum_{i=2}^{12} i f_X(i) = 7.$$

Note that $\mathscr{E}[Y]$ is not one of the possible values of Y. ////

EXAMPLE 11 Let X be a continuous random variable with probability density function $f_X(x) = \lambda e^{-\lambda x} I_{[0,\infty)}(x)$.

$$\mathscr{E}[X] = \int_{-\infty}^{\infty} x f_X(x)\, dx = \int_0^{\infty} x \lambda e^{-\lambda x}\, dx = \frac{1}{\lambda}.$$

The corresponding cumulative distribution function is

$$F_X(x) = (1 - e^{-\lambda x}) I_{[0,\infty)}(x); \text{ so } \mathscr{E}[X] = \int_0^{\infty} [1 - F_X(x)]\, dx$$
$$- \int_{-\infty}^{0} F_X(x)\, dx = \int_0^{\infty} (1 - 1 + e^{-\lambda x})\, dx = 1/\lambda. \qquad ////$$

EXAMPLE 12 Let X be a random variable with cumulative distribution function given by $F_X(x) = (1 - pe^{-\lambda x})I_{[0,\infty)}(x)$; then

$$\mathscr{E}[X] = \int_0^\infty [1 - F_X(x)]\,dx - \int_{-\infty}^0 F_X(x)\,dx = \int_0^\infty pe^{-\lambda x}\,dx = \frac{p}{\lambda}.$$

Here, we have used Eq. (6) to find the mean of a random variable that is partly discrete and partly continuous. ////

EXAMPLE 13 Let X be a random variable with probability density function given by $f_X(x) = x^{-2}I_{[1,\infty)}(x)$; then

$$\mathscr{E}[X] = \int_1^\infty x\,\frac{dx}{x^2} = \lim_{b\to\infty} \log_e b = \infty,$$

so we say that $\mathscr{E}[X]$ does not exist. We might also say that the mean of X is infinite since it is clear here that the integral that defines the mean is infinite. ////

4.2 Variance

The mean of a random variable X, defined in the previous subsection, was a measure of *central location* of the density of X. The *variance* of a random variable X will be a measure of the *spread* or *dispersion* of the density of X.

Definition 11 Variance Let X be a random variable, and let μ_X be $\mathscr{E}[X]$. The *variance of* X, denoted by σ_X^2 or var $[X]$, is defined by

(i) $$\text{var}\,[X] = \sum_j (x_j - \mu_X)^2 f_X(x_j) \tag{7}$$

if X is discrete with mass points $x_1, x_2, \ldots, x_j, \ldots$.

(ii) $$\text{var}\,[X] = \int_{-\infty}^\infty (x - \mu_X)^2 f_X(x)\,dx \tag{8}$$

if X is continuous with probability density function $f_X(x)$.

(iii) $$\text{var}\,[X] = \int_0^\infty 2x[1 - F_X(x) + F_X(-x)]\,dx - \mu_X^2 \tag{9}$$