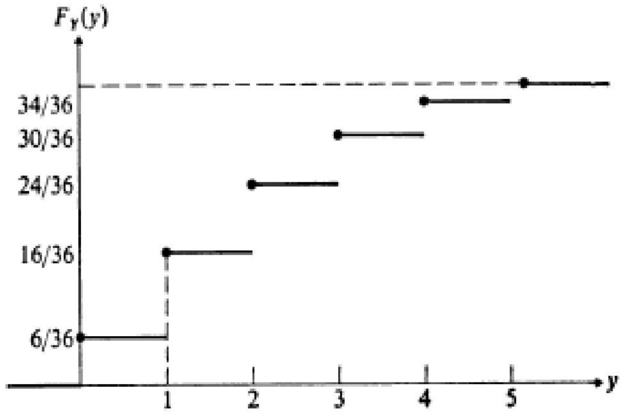

FIGURE 2

Properties of a Cumulative Distribution Function $F_X(\cdot)$

(i) $F_X(-\infty) \equiv \lim_{x \to -\infty} F_X(x) = 0$, and $F_X(+\infty) \equiv \lim_{x \to +\infty} F_X(x) = 1$.

(ii) $F_X(\cdot)$ is a monotone, nondecreasing function; that is, $F_X(a) \leq F_X(b)$ for $a < b$.

(iii) $F_X(\cdot)$ is continuous from the right; that is,

$$\lim_{0 < h \to 0} F_X(x + h) = F_X(x).$$

Except for (ii), we will not prove these properties. Note that the event $\{\omega\colon X(\omega) \leq b\} = \{X \leq b\} = \{X \leq a\} \cup \{a < X \leq b\}$ and $\{X \leq a\} \cap \{a < X \leq b\} = \phi$; hence, $F_X(b) = P[X \leq b] = P[X \leq a] + P[a < X \leq b] \geq P[X \leq a] = F_X(a)$ which proves (ii). Property (iii), the continuity of $F_X(\cdot)$ from the right, results from our defining $F_X(x)$ to be $P[X \leq x]$. If we had defined, as some authors do, $F_X(x)$ to be $P[X < x]$, then $F_X(\cdot)$ would have been continuous from the left.

3 DENSITY FUNCTIONS

Random variable and the cumulative distribution function of a random variable have been defined. The cumulative distribution function described the distribution of values of the random variable. For two distinct classes of random variables, the distribution of values can be described more simply by using *density functions*. These two classes, distinguished by the words "discrete" and "continuous," are considered in the next two subsections.

--(4)

3.1 Discrete Random Variables

Definition 4 Discrete random variable A random variable X will be defined to be *discrete* if the range of X is countable. If a random variable X is discrete, then its corresponding cumulative distribution function $F_X(\cdot)$ will be defined to be *discrete*. ////

By the range of X being countable we mean that there exists a finite or denumerable set of real numbers, say $x_1, x_2, x_3, \ldots$, such that X takes on values only in that set. If X is discrete with distinct values $x_1, x_2, \ldots, x_n, \ldots$, then $\Omega = \bigcup_n \{\omega\colon X(\omega) = x_n\} = \bigcup_n \{X = x_n\}$, and $\{X = x_i\} \cap \{X = x_j\} = \phi$ for $i \neq j$; hence $1 = P[\Omega] = \sum_n P[X = x_n]$ by the third axiom of probability.

Definition 5 Discrete density function of a discrete random variable If X is a discrete random variable with distinct values $x_1, x_2, \ldots, x_n, \ldots$, then the function, denoted by $f_X(\cdot)$ and defined by

$$f_X(x) = \begin{cases} P[X = x_j] & \text{if } x = x_j, j = 1, 2, \ldots, n, \ldots \\ 0 & \text{if } x \neq x_j \end{cases} \tag{1}$$

is defined to be the *discrete density function* of X. ////

The values of a discrete random variable are often called *mass points*; and, $f_X(x_j)$ denotes the *mass* associated with the *mass point* x_j. *Probability mass function*, *discrete frequency function*, and *probability function* are other terms used in place of *discrete density function*. Also, the notation $p_X(\cdot)$ is sometimes used intead of $f_X(\cdot)$ for discrete density functions. $f_X(\cdot)$ is a function with domain the real line and counterdomain the interval [0, 1]. If we use the indicator function,

$$f_X(x) = \sum_{n=1}^{\infty} P[X = x_n] I_{\{x_n\}}(x), \tag{2}$$

where $I_{\{x_n\}}(x) = 1$ if $x = x_n$ and $I_{\{x_n\}}(x) = 0$ if $x \neq x_n$.

Theorem 1 Let X be a discrete random variable. $F_X(\cdot)$ can be obtained from $f_X(\cdot)$, and vice versa.