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2 RANDOM VARIABLE AND CUMULATIVE DISTRIBUTION FUNCTION

2.2 Definitions

We commence by defining a random variable.

Definition 1 Random Variable For a given probability space $(\Omega, \mathscr{A}, P[\cdot])$, a *random variable*, denoted by X or $X(\cdot)$, is a function with domain Ω and counterdomain the real line. The function $X(\cdot)$ must be such that the set A_r, defined by $A_r = \{\omega: X(\omega) \leq r\}$, belongs to $\mathscr{A}$ for every real number r. ////

EXAMPLE 1 Consider the experiment of tossing a single coin. Let the random variable X denote the number of heads. $\Omega = \{\text{head, tail}\}$, and $X(\omega) = 1$ if $\omega = \text{head}$, and $X(\omega) = 0$ if $\omega = \text{tail}$; so, the random variable X associates a real number with each outcome of the experiment. We called X a random variable so mathematically speaking we should show

that it satisfies the definition; that is, we should show that $\{\omega: X(\omega) \leq r\}$ belongs to $\mathscr{A}$ for every real number r. $\mathscr{A}$ consists of the four subsets: ϕ, {head}, {tail}, and Ω. Now, if $r < 0$, $\{\omega: X(\omega) \leq r\} = \phi$; and if $0 \leq r < 1$, $\{\omega: X(\omega) \leq r\} = \{\text{tail}\}$; and if $r \geq 1$, $\{\omega: X(\omega) \leq r\} = \Omega = \{\text{head, tail}\}$. Hence, for each r the set $\{\omega: X(\omega) \leq r\}$ belongs to $\mathscr{A}$; so $X(\cdot)$ is a random variable. ////

EXAMPLE 2 Consider the experiment of tossing two dice. Ω can be described by the 36 points displayed in Fig. 1. $\Omega = \{(i, j): i = 1, \ldots, 6 \text{ and } j = 1, \ldots, 6\}$. Several random variables can be defined; for instance, let X denote the sum of the upturned faces; so $X(\omega) = i + j$ if $\omega = (i, j)$. Also, let Y denote the absolute difference between the upturned faces; then $Y(\omega) = |i - j|$ if $\omega = (i, j)$. It can be shown that both X and Y are random variables. We see that X can take on the values 2, 3, ..., 12 and Y can take on the values 0, 1, ..., 5. ////

Definition 2 Cumulative distribution function The *cumulative distribution function* of a random variable X, denoted by $F_X(\cdot)$, is defined to be that function with domain the real line and counterdomain the interval

[0, 1] which satisfies $F_X(x) = P[X \leq x] = P[\{\omega: X(\omega) \leq x\}]$ for every real number x. ////

The use of each of the three words in the expression "cumulative distribution function" is justifiable. A cumulative distribution function is first of all a *function*; it is a *distribution* function inasmuch as it tells us how the values of the random variable are distributed, and it is a *cumulative* distribution function since it gives the distribution of values in cumulative form. Many writers omit the word "cumulative" in this definition. Examples and properties of cumulative distribution functions follow.

EXAMPLE 3 Consider again the experiment of tossing a single coin. Assume that the coin is fair. Let X denote the number of heads. Then,

$$F_X(x) = \begin{cases} 0 & \text{if } x < 0 \\ \frac{1}{2} & \text{if } 0 \le x < 1 \\ 1 & \text{if } 1 \le x. \end{cases}$$

Or $F_X(x) = \frac{1}{2}I_{[0,1)}(x) + I_{[1,\infty)}(x)$ in our indicator function notation. ////

EXAMPLE 4 In the experiment of tossing two fair dice, let Y denote the absolute difference. The cumulative distribution of Y, $F_Y(\cdot)$, is sketched in Fig. 2. Also, let X_k denote the value on the upturned face of the kth die for $k = 1, 2$. X_1 and X_2 are different random variables, yet both have the same cumulative distribution function, which is $F_{X_k}(x) = \sum_{i=1}^{5} \frac{i}{6} I_{[i,i+1)}(x) + I_{[6,\infty)}(x)$ and is sketched in Fig. 3. ////

Careful scrutiny of the definition and above examples might indicate the following properties of any cumulative distribution function $F_Y(\cdot)$.

Definition **Indicator function** Let Ω be any space with points ω and A any subset of Ω. The *indicator function* of A, denoted by $I_A(\cdot)$, is the function with domain Ω and counterdomain equal to the set consisting of the two real numbers 0 and 1 defined by

$$I_A(\omega) = \begin{cases} 1 & \text{if } \omega \in A \\ 0 & \text{if } \omega \notin A. \end{cases}$$

$I_A(\cdot)$ clearly "indicates" the set A. ////

Properties of Indicator Functions Let Ω be any space and $\mathscr{A}$ any collection of subsets of Ω:

(i) $I_A(\omega) = 1 - I_{\bar{A}}(\omega)$ for every $A \in \mathscr{A}$.

(ii) $I_{A_1A_2\cdots A_n}(\omega) = I_{A_1}(\omega) \cdot I_{A_2}(\omega) \cdots I_{A_n}(\omega)$ for $A_1, \ldots, A_n \in \mathscr{A}$.

(iii) $I_{A_1 \cup A_2 \cup \cdots \cup A_n}(\omega) = \max[I_{A_1}(\omega), I_{A_2}(\omega), \ldots, I_{A_n}(\omega)]$ for $A_1, \ldots, A_n \in \mathscr{A}$.

(iv) $I_A^2(\omega) = I_A(\omega)$ for every $A \in \mathscr{A}$.