3.2 Continuous Random Variables

Definition 7 Continuous random variable A random variable X is called *continuous* if there exists a function $f_X(\cdot)$ such that $F_X(x) = \int_{-\infty}^{x} f_X(u)du$ for every real number x. The cumulative distribution function $F_X(\cdot)$ of a continuous random variable X is called *absolutely continuous.* ////

Definition 8 Probability density function of a continuous random variable If X is a continuous random variable, the function $f_X(\cdot)$ in $F_X(x) = \int_{-\infty}^{x} f_X(u)\,du$ is called the *probability density function* of X. ////

Other names that are used instead of probability density function include *density function*, *continuous density function*, and *integrating density function*.

Note that strictly speaking *the* probability density function $f_X(\cdot)$ of a random variable X is not uniquely defined. All that the definition requires is that the integral of $f_X(\cdot)$ gives $F_X(x)$ for every x, and more than one function $f_X(\cdot)$ may satisfy such requirement. For example, suppose $F_X(x) = xI_{[0,1)}(x) + I_{[1,\infty)}(x)$; then $f_X(u) = I_{(0,1)}(u)$ satisfies $F_X(x) = \int_{-\infty}^{x} f_X(u)\,du$ for every x, and so $f_X(\cdot)$ is a probability density function of X. However $f_X(u) = I_{(0,\frac{1}{2})}(u) + 69I_{\{\frac{1}{2}\}}(u) + I_{(\frac{1}{2},1)}(u)$ also satisfies $F_X(x) = \int_{-\infty}^{x} f_X(u)\,du$. (The idea is that if the value of a function is changed at only a "few" points, then its integral is unchanged.) In practice a unique choice of $f_X(\cdot)$ is often dictated by continuity considerations and for this reason we will usually allow ourselves the liberty of speaking of *the* probability density when in fact *a* probability density is more correct.

Theorem 2 Let X be a continuous random variable. Then $F_X(\cdot)$ can be obtained from an $f_X(\cdot)$, and vice versa.

telephone conversation. One could model this experiment by assuming that the distribution of X is given by $F_X(x) = (1 - e^{-\lambda x})I_{[0,\infty)}(x)$, where λ is some positive number. The corresponding probability density function would be given by $f_X(x) = \lambda e^{-\lambda x}I_{[0,\infty)}(x)$. If we assume that telephone conversations are measured in minutes, $P[5 < X \le 10] = \int_5^{10} \lambda e^{-x\lambda}\,dx = e^{-5\lambda} - e^{-10\lambda} = e^{-1} - e^{-2} \approx .23$ for $\lambda = \frac{1}{5}$, or $P[5 < X \le 10] = P[X \le 10] - P[X \le 5] = (1 - e^{-\lambda 10}) - (1 - e^{-\lambda 5}) = e^{-1} - e^{-2}$ for $\lambda = \frac{1}{5}$. ////

The probability density function is used to calculate the probability of events defined in terms of the corresponding continuous random variable X. For example, $P[a < X \le b] = \int_a^b f_X(x)\,dx$ for $a < b$.

---(7)

Definition 9 Probability density function Any function $f(\cdot)$ with domain the real line and counterdomain $[0, \infty)$ is defined to be a *probability density function* if and only if

(i) $f(x) \geq 0$ for all x.

(ii) $\int_{-\infty}^{\infty} f(x)\,dx = 1.$ ////

With this definition we can speak of probability density functions without reference to random variables. We might note that a probability density function of a continuous random variable as defined in Definition 8 does indeed possess the two properties in the above definition.

Reference

Introduction to the Theory of Statistics

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