Chapter Seven: Einstein special theory of relativity

7.1 Pre- relativistic physics

The starting point for our work is Newton's laws of motion. These can be stated as follows:

- ***Free particles move with constant velocity.***
- ***The vector force F is proportional to the rate of change of momentum i.e.***

  $F = \frac{d}{dt}(m\upsilon)$

- ***To every action there is an equal and opposite reaction.***

The first of these laws singles out inertial frames as the non- accelerating ones. Consider now a frame *O* [i.e. a set of spatial coordinates (*x*, *y*, *z*) and a time coordinate *t*], and another frame O' with coordinates (x', y', z', t') which moves in the *x* direction with uniform speed υ relative to the frame *O*.

Common sense suggests that the two sets of coordinates are related by

$$\begin{aligned} x' &= x - \upsilon t, \\ y' &= y, \\ z' &= z, \\ t' &= t \end{aligned} \qquad (7.1)$$

These are the Galilean transformations.

Figure 7.1: I do not know what I may appear to the world; but to myself I seem to have been only like a boy playing on the sea- shore, and diverting myself in now and then finding a smoother pebble or a prettier shell than ordinary, whilst the great ocean of truth lay all undiscovered before me. *Isaac Newton.*

If the particle has a velocity (u) with components (u_1, u_2, u_3) in *O*, its velocity in O' is:

$$u'_1 = u_1 - v,$$
$$u'_2 = u_2\,,$$
$$u_3 = u_3\,, \qquad \textbf{(7.2)}$$

where

$$u_1 = \frac{dx_1}{dt}, \quad u_2 = \frac{dx_2}{dt}, \quad u_3 = \frac{dx_3}{dt} \qquad \textbf{(7.3)}$$

7.2 The equations of electromagnetism

Maxwell's equations for the electromagnetic field [in units with $\varepsilon_o = \mu_o = c = 1$] are:

$$\vec{\nabla} \times \vec{B} - \frac{\partial E}{\partial t} = 4\pi j, \qquad \vec{\nabla} \times \vec{E} - \frac{\partial \vec{B}}{\partial t} = 0,$$
$$\vec{\nabla} \cdot \vec{E} = 4\pi\rho \quad , \qquad \vec{\nabla} \cdot \vec{B} = 0 \qquad (7.4)$$

These equations show that one can have electromagnetic waves in a vacuum which travel with a speed $c = 3\times10^8$ m/s, thus light is a form of electromagnetic radiation.

Figure 7.2: **James Clerk Maxwell**

Ether was postulated as the medium which carries these waves [i.e. the speed *c* is the speed with respect to the ether] and naturally this was identified with the absolute space of Newton.

Attempts to measure the ether drift by Michelson and Morley in 1887 gave a null result, so effects like stellar aberration could not be due to the earth dragging the ether.

Another great problem was that Maxwell's equations did not appear to obey the principle of Galilean Relativity i.e. they were not invariant under the Galilean transformations. This means that in a moving space ship the electric and optical phenomena should be different from those in a stationary ship!!! Thus one could use for example optical phenomena to determine the speed of the ship.

One of the consequences of Maxwell's equations is that if there is a disturbance in the field such that light is generated, these electromagnetic waves go out in all directions equally and at the same speed *c*. Another consequence of the equations is that if the source of the disturbance is moving, the light emitted goes through space at the same speed *c*. This is analogous to the case of sound being likewise independent of the motion of the source. This independence of the motion of the source in the case of light brings up an interesting problem.

Suppose we are riding in a car that is going at a speed *u*, and light from the rear is going past the car with speed *c*. According to the Galilean transformations, the apparent speed of the passing light, as we measure it in the car, should not be c, but *c-u*. This means that by measuring the speed of the light going past the car, one could determine the absolute speed of the car. This is crazy because we already know that the laws of mechanics do not permit the measurement of absolute velocity.

- **Something is wrong with the laws of Physics!!**

At first people thought that it was Maxwell's equations that were at fault and they modified them so that they were invariant under the Galilean

transformations; but this led to the prediction of new electrical phenomena which could not be found experimentally.

Then in 1903 Lorentz made a remarkable and curious discovery. He found that when he applied the following transformations to Maxwell's equations they remained invariant.

$$
\begin{aligned}
t' &= \frac{t - \upsilon x / c^2}{\sqrt{1-\upsilon^2/c^2}}, \\
x' &= \frac{x - \upsilon t}{\sqrt{1-\upsilon^2/c^2}}, \\
y' &= y \\
z' &= z,
\end{aligned}
\qquad (7.5)
$$

These are the famous Lorentz transformations.

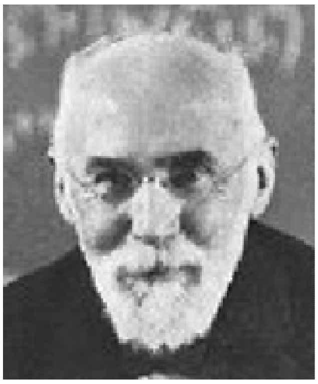

Figure 7.3: Hendrik Antoon Lorentz.

7.3 The principle of Special Relativity

In 1905 Einstein generalized the Galilean relativity principle [applicable only to Newton's laws] to the whole of Physics by postulating that:

1. ***All inertial frames are equivalent for all experiments i.e. no experiment can measure absolute velocity.***
2. ***Maxwell's equations and the speed of light must be the same for all observers.***

Einstein's motivation was to avoid inconsistencies between Maxwell's equations and Galilean relativity. The Lorentz transformations must relate to actual space and time measurements.

Special Relativity abolishes the idea of absolute space [for example the ether] and absolute time, but it leaves unexplained the origin of inertial frames. Mach's principle says that the inertial frames are determined be the rest of the matter in the universe [i.e. those that are non- accelerating with respect to the rest of the universe]. Einstein tried later to incorporate this idea into General Relativity.

7.4 Time dilation

Suppose we have someone in a spaceship O' moving at some velocity *v* relative to an observer *O*. The guy in the spaceship turns his stereo on and starts listening to some rap music!! With a clock he measures the time interval between two beats $\Delta t' = t'_B - t'_A$. What time interval does the observer *O* measure? To calculate this it is easier to use the inverse Lorentz transformations. By the Relativity principle we have:

$$t = \gamma(t' + vx'/c^2) \qquad \textbf{(7.6)}$$

so

$$t_B - t_A = \gamma\left[t'_B - t'_A + \frac{v^2}{c^2}(x'_B - x'_A)\right] \qquad (7.7)$$

But the stereo is stationary relative to O' so $x'_A = x'_B$, so we end up with:

$$\Delta t = t_B - t_A = \gamma\Delta t' = \frac{\Delta t'}{\sqrt{1-\frac{v^2}{c^2}}} \qquad (7.8)$$

Since $\Delta t > \Delta t'$, time in the spaceship slows down! That is, when the clock in the spaceship records 1 second elapsed, as seen be the man in the ship, it shows $1/\sqrt{1-v^2/c^2}$ seconds to the man outside.

The slowing of clocks in a moving system is a real effect and it applies equally to all kinds of time, for example biological, chemical reaction rates, even to the rate of growth of a cancer in a cancer patient! How is this so you may ask? If the rate of growth of the cancer was the same for a stationary patient as for a moving one, it would be possible to use the rate of cancer development to determine the speed of the ship!

A very interesting example of the slowing down of time is that of the cosmic ray muons. These are particles that disintegrate spontaneously after an average lifetime of about 2.2×10^{-6} seconds. It is clear that in its short lifetime a muon cannot, even at the speed of light, travel more than 600 m. But although the muons are created at the top of the atmosphere, some 10 km up, we can detect them down here on earth. How can that be!!? From the muon's point of view (i.e. from their frame of reference) they only live about 2μs. However

from our point of view they live considerably longer, indeed long enough to reach the surface of the earth (by a factor of $1/\sqrt{1-v^2/c^2}$).

7.6 Length contraction

Now suppose the observer *O* wants to measure the length of the spaceship. He can only do this by making an instantaneous measurement of the spatial coordinates of the end of the ship i.e. x_A and x_B.

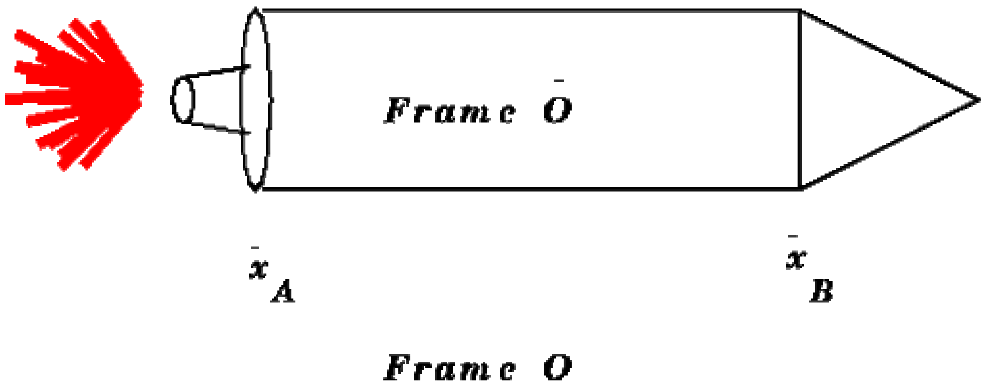

Figure 7.8: **Length contraction**

The Lorentz transformations give:

$$\begin{aligned} x_1' &= \gamma(x_1 - vt_1) \\ x_2' &= \gamma(x_2 - vt_2) \end{aligned} \qquad (7.9)$$

so the length is

$$\begin{aligned} L &= x_2 - x_1 \\ &= \gamma^{-1}(x_2' - x_1') \\ &= \gamma^{-1}L' \end{aligned} \qquad (7.10)$$

since $t_1 = t_2$ [instantaneous measurement by *O*]. Writing this result in terms of v and *c* we have:

$$L = \sqrt{1 - v^2/c^2}\; L' \qquad (7.11)$$

Note that this is not a physical effect on the rod but an effect of space-time itself.

7.7 The twin paradox

To continue our discussion of the Lorentz transformations and relativistic effects, we consider the famous ``twin paradox'' of Peter and Paul. When they are old enough to drive a spaceship, Paul flies away from earth at very high speed. Because Peter, who is left on the ground, sees Paul going so fast, all of Pauls' clocks appear to go slower, from Peter's point of view. Of course, Paul notices nothing unusual. After a while he returns and finds that he is younger that Peter! Now some people might say ``heh, heh, heh, from the point of view of Paul, can't we say that Peter was moving and should therefore appear to age more slowly? By symmetry, the only possible result is that they are both the same age when they meet''. But in order for them to come back together and make a comparison, Paul must turn around which involves decelerating and accelerating and during that period he is not in an inertial frame. This breaks the apparent symmetry and so resolves the paradox.

Example: 1

(a) An observer on Earth sees a spaceship at an altitude of 4350 km moving downward toward Earth with a speed of 0.970*c*. What is the distance from the spaceship to Earth as measured by the spaceship's captain?

(b) After firing his engines, the captain measures her ship's altitude as 267 km, while the observer on Earth measures it to be 625 km. What is the speed of the spaceship at this instant?

Solution

(a) Find the distance from the ship to Earth as measured by the captain.

Substitute into Equation 7.11, getting the altitude as measured by the captain in the ship.

$$L = \sqrt{1-\frac{v^2}{c^2}}\, L' = \sqrt{1-\frac{(0.970c)^2}{c^2}}\,(4350\,\mathrm{km}) = 1.06\times 10^3\ \mathrm{km}$$

(b) What is the subsequent speed of the spaceship if the Earth observer measures the distance from the ship to Earth as 625 km and the captain measures it as 267 km?

$$L = \sqrt{1-\frac{v^2}{c^2}}\ \overline{L} \Rightarrow \quad L^2 = \left(1-\frac{v^2}{c^2}\right)\overline{L}^2 \Rightarrow 1-\frac{v^2}{c^2} = \left(\frac{L}{\overline{L}}\right)^2$$

$$v = c\sqrt{1-\left(L/\overline{L}\right)^2} = c\sqrt{1-\left(267\mathrm{km}/625\mathrm{km}\right)^2} = 0.904\,c$$

7.8 Relativistic momentum and relativistic energy

Properly describing the motion of particles within the framework of special relativity requires generalizing Newton's laws of motion and the definitions of momentum and energy. These generalized definitions reduce to the classical (nonrelativistic) definitions when υ is much less than *c*.

p=mυ (for classical physics)

$$p = \frac{m_o \upsilon}{\sqrt{1-\frac{\upsilon^2}{c^2}}} = \gamma m_o \upsilon \qquad \textbf{(7.12)}$$

We have seen that the definition of momentum required generalization to make it compatible with the principle of relativity. Likewise, the definition of kinetic energy requires modification in relativistic mechanics. Einstein found that the correct expression for the kinetic energy of an object is

$$KE = \gamma m_o c^2 - m_o c^2 \qquad \textbf{(7.13)}$$

The constant term $m_o c^2$ in Equation 7.13, which is independent of the speed of the object, is called the rest energy of the object, E_R:

$$E_R = m_o c^2$$

The term $\gamma m_o c^2$ in Equation 36 depends on the object's speed and is the sum of the kinetic and rest energies. We define $\gamma m_o c^2$ to be the total energy E, so that

total energy = kinetic energy +rest energy

or, using Equation 7.13,

$$E = KE + m_o c^2 = \gamma m_o c^2$$

Example: 2

An electron moves with a speed υ = 0.850c. Find its total energy and kinetic energy in mega electron volts (MeV), and compare the latter to the classical kinetic energy (10^6 eV= 1 MeV).

Solution:

$$E = \frac{m_e c^2}{\sqrt{1-\frac{\upsilon^2}{c^2}}} = \frac{(9.11\times10^{-31}\,\text{kg})(3\times10^{8}\,\text{m/s})^2}{\sqrt{1-(0.850\text{c/c})^2}} = 1.56\times10^{-13}\,\text{J} = 0.975\,\text{MeV}$$

$$KE = E - m_e c^2 = 0.975\,\text{MeV} - 0.511\,\text{MeV} = 0.464\,\text{MeV}$$

$$KE_{classical} = \frac{1}{2} m_e \upsilon^2$$

$$= \tfrac{1}{2}(9.11\times10^{-31}\,\text{kg})(0.850\times3\times10^{8}\,\text{m/s})^2$$

$$= 2.96\times10^{-14}\ \text{J} = 0.185\ \text{MeV}$$