Chapter Three: Wave particle duality

3.1 De Broglie hypothesis

An important step in the development of a satisfactory quantum theory occurred when Louis de Broglie in 1929 in his PhD work postulated that:

- **Nature is strikingly symmetrical.**
- **Our universe is composed entirely of light and matter.**
- **If light has a dual wave-particle nature, perhaps matter also has this nature.**

The dual nature of light shows up in equations;

$$\lambda = \frac{h}{p} \ , \qquad E = h\nu$$

Each equation contains within its structure both a wave concept $(\lambda; \nu)$, and a particle $(p; E)$.
The photon also has an energy given by the relationship from the relativity theory

$$E = mc^2 \ \text{(for photon)} \ , \ E = \frac{p^2}{2m} \ \ \text{(for particles)}$$

Since $E = h\nu = \dfrac{hc}{\lambda}$ **, we find the wavelength**

$$\lambda = \frac{h}{mc} = \frac{h}{p} \ ,$$

where p is the momentum of the photon.

This does not mean that light has mass, but because mass and energy can be interconverted, it has an energy that is equivalent to some mass. De Broglie postulated that a particle can have a wave character, and predicted that the wavelength of a matter wave would also be given by the same equation that held for light, where now p would be the momentum of the particle

$$\lambda = \frac{h}{m\nu} = \frac{h}{p}$$

where v is the velocity of the particle.

Example: **What is the de Broglie wavelength of an electron whose kinetic energy is   K = 100 eV?**

Calculate the velocity of the electron from which we then find the de Broglie wavelength corresponding to that velocity.

The velocity of the electron of energy 100 eV is

$$v = \sqrt{\frac{2K}{m}} = 5.9 \times 10^6 \quad m/s$$

**Hence, the de Broglie wavelength corresponding to this velocity is**

$$\lambda = \frac{h}{mv} = \frac{6.63 \times 10^{-34} \, J.s}{(9.11 \times 10^{-31} \, Kg)(5.9 \times 10^6 \, m)} = 1.2 \ \overset{o}{A}$$

3.2 Heisenberg uncertainty principle

In *quantum mechanics*, the *Heisenberg uncertainty principle* states that certain pairs of physical properties, like position and momentum, cannot both be known to arbitrary precision. That is, the more precisely one property is known, the less precisely the other can be known. According to the uncertainty principle, it is, for instance, impossible to measure simultaneously both position and velocity of a microscopic particle with any degree of accuracy or certainty.

$$\Delta x \, \Delta p \geq \frac{\eta}{2}$$

$$\Delta E \, \Delta t \geq \frac{\eta}{2}$$
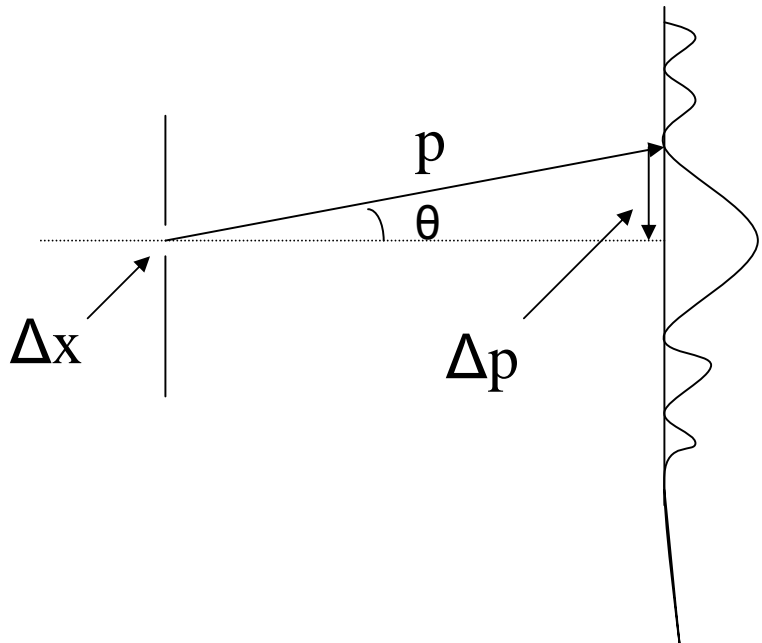
3.3 Single slit experiment

In this section we will discuss the nature of the Fraunhofer diffraction pattern produced by a single slit. Some important features of this problem can be deduced by examining waves coming from various portions of the slit, as shown in Figure (3.1). According to Huygens principle, each portion of the slit acts as a source of waves. Hence, light from one portion of the slit can interfere with light from another portion, and the resultant intensity on the screen will depend on the angle θ.
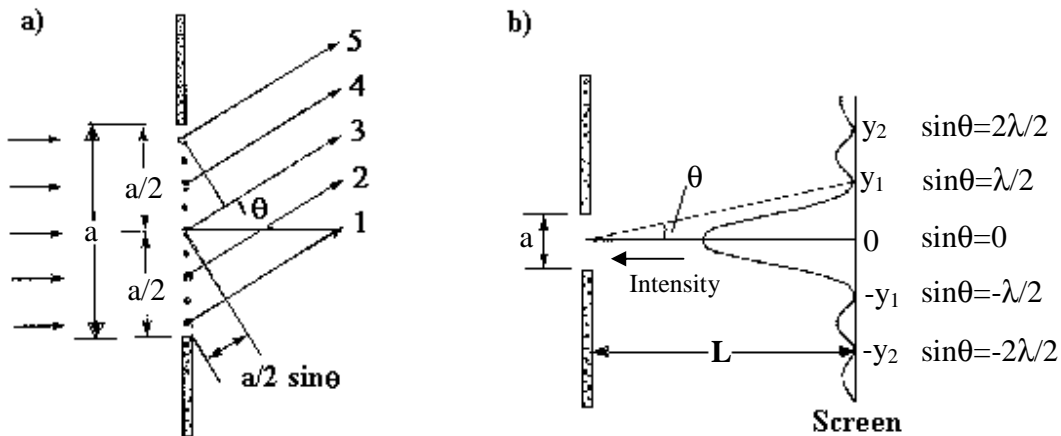

Figure 3.1: Diffraction by single narrow slit and its intensity distribution

To analyze the diffraction pattern, it is convenient to divide the slit in two halves, as in Figure (3.1). All the waves that originate from the slit are in phase. Consider waves 1 and 3, which originate from the bottom and center of the slit, respectively. Wave 1 travels farther than wave 3 by an amount equal to the path difference $(a/2)\sin\theta$, where a is the width of the slit. Similarly, the path difference between waves 2 and 4 is also $(a/2)\sin\theta$. If this path difference is exactly one half of a wavelength $(\lambda/2)$, the two waves cancel each other and destructive interference results. This is true, in fact, for any two waves that originate at points separated by half the slit width because the phase difference between two such points is 180°. Therefore, waves from the upper half of the slit interfere *destructively* with waves from the lower half of the slit when

$$\frac{a}{2}\sin\theta = \frac{\lambda}{2}$$

or when

$$\sin\theta = \frac{\lambda}{a}$$

If we divide the slit into four parts rather than two and use similar reasoning, we find that the screen is also dark when

$$\sin\theta = \frac{2\lambda}{a}$$

3.4 Young's double-slit experiment

Young originally carried out his double-slit experiment to test the wave nature of light and his simple arrangement still finds practical use in measurements of the coherence of light. Figure (3.2) shows a typical experimental layout. Waves propagate from the source slit S through the two slits, Σ_1 and Σ_2, to a point P in the detection plane equals the sum of the electric field amplitudes that arrive at that point via slits Σ_1 and Σ_2. In any interference or diffraction calculation the resultant amplitude at the final point is determined by summing the contributions from all possible paths taking account of the phase. For the double slits
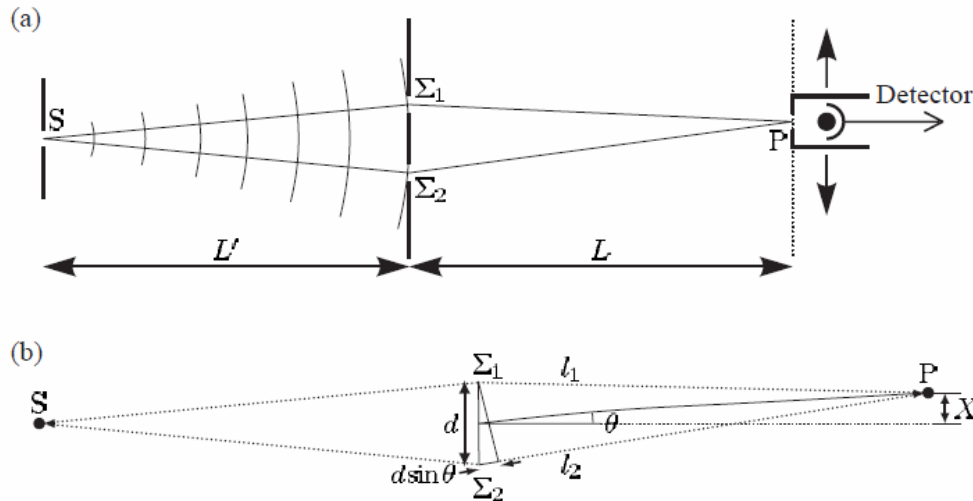

Figure 3.2: The Young's double slit experiment

we define λ_1 as the distance from S to P via Σ_1, and similarly λ_2 for the path through slit Σ_2, as shown by the dotted lines in Fig. 3.2 (b). Slits of the same size contribute equally to the total amplitude at a point on the plane P:

$$E_p \alpha E_o \left(e^{-i2\pi\lambda_1/\lambda} + e^{-i2\pi\lambda_2/\lambda} \right)$$

The intensity is proportional to the square of this amplitude, $I\alpha|E|^2$ so that

$$I = I_0 \cos^2\left(\frac{\phi}{2}\right)$$

Here $\phi = 2\pi(\lambda_2 - \lambda_1)/\lambda$ is the phase difference between the two arms and I_0 is the maximum intensity. Bright fringes occur at positions in the detection plane where the contributions from the two paths interfere constructively; these correspond to $\phi = 2n\pi$, with n an integer, or equivalently

$$\lambda_2 - \lambda_1 = n\lambda$$

To find the spacing of the fringes in the plane of observation we define the coordinate **X** measured perpendicular to the long axis of the slits in the plane **P**. In terms of the small angle defined in Fig. 3.2 (b) this becomes

$$X = L\tan\theta$$

A similar small angle approximation allows us to express the path length difference in Fig. 3.2 (b) as

$$\lambda_2 - \lambda_1 = \Delta\lambda + d\sin\theta$$

Here $\Delta\lambda = \overline{S\Sigma_1} - \overline{S\Sigma_2}$ the path difference before the slits. The path length difference from the two slits of separation d to *P* is *d* sinθ. The last three equations give the spacing of the fringes as

$$\Delta X = \frac{L\lambda}{d}$$

3.5   Phase and group velocities

We know matter waves (de Broglie waves)

$$\lambda = \frac{h}{p} = \frac{h}{m\text{v}}$$

where v is the velocity of a particle of the mass *m*.

For particles and according to relativity

$$E = mc^2 \text{ and since } E = h\nu \text{ then}$$

$$mc^2 = h\nu \rightarrow \nu = \frac{mc^2}{h}$$

If we define u as the phase velocity and is given by

$$u = \lambda\nu = \frac{h}{m\text{v}}\frac{mc^2}{h} = \frac{c^2}{\text{v}}$$

Thus, the velocity of the matter waves is larger than speed of light in vacuum, u > c, unless v > c.

This result seems disturbing because it appears that the matter waves would propagate faster than the speed of light and would not be able to keep up with particles whose motion they govern.

However, the phase velocity of a wave is the velocity of the wave front, not its amplitude. The maximum of the amplitude of a given wave can propagate at different velocity, called group velocity. At this velocity the energy (information) is transmitted. Usually, $v_g = u$, but in the case of dispersion, u(ν), the group velocity $v_g < u$. Thus,

the matter wave should be dispersive to match the requirement of $v_g < u$ when $u > c$.

The group velocity is given by

$$V_g = \frac{d\omega}{d\kappa}$$

where ω is the angular frequency and κ is the wave number.

Phase Velocity (u)
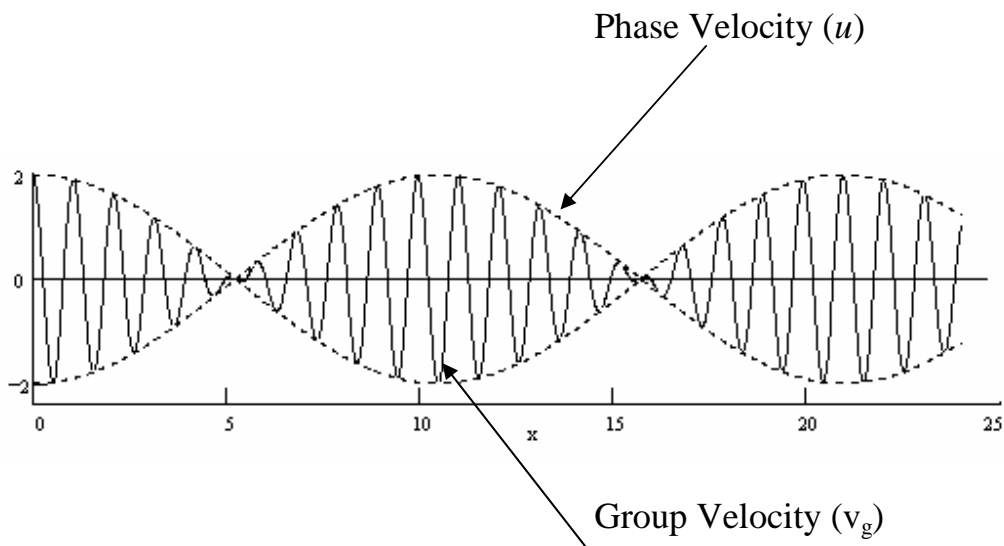
Group Velocity (v_g)

since $\omega = 2\pi\nu$ and $\kappa = \frac{2\pi}{\lambda}$ then we get

$$u = \lambda\nu = \frac{2\pi}{\kappa}\frac{\omega}{2\pi} = \frac{\omega}{\kappa} \Rightarrow u = \frac{\omega}{\kappa}$$

Question:

The dispersion relation for free relativistic electron waves is

$$\omega = \sqrt{c^2\kappa^2 + \left(mc^2/\eta\right)^2}$$

(a) Calculate expressions for the phase velocity u and group velocity v_g of these waves and show that their product is constant, independent of κ.

(b) From the result (a), what can you conclude about v_g if u > c?

Solution:

      (a)   From the definition of the phase velocity

$$u = \frac{\omega}{\kappa} = \sqrt{c^2 + \left(\frac{mc^2}{\kappa h}\right)^2}$$

We see that the phase velocity u > c.

From the definition of the group velocity

$$v_g = \frac{d\omega}{d\kappa} = \frac{1}{2}\frac{2c^2\kappa}{\sqrt{c^2\kappa^2 + \left(mc^2/h\right)^2}}$$

$$= \frac{c^2\kappa}{c\kappa\sqrt{1 + \left(mc/\kappa h\right)^2}} = \frac{c}{\sqrt{1 + \left(mc/\kappa h\right)^2}}$$

Thus, the group velocity v_g < c.

      However, the product

$$uv_g = \sqrt{c^2 + \left(\frac{mc^2}{\kappa h}\right)^2} \times \frac{c}{\sqrt{1 + \left(mc/\kappa h\right)^2}} = c^2$$

      is constant independent of κ.

(b) We see from (a) that in general for dispersive waves for which u > c, the group velocity v_g < c. Only when u = c, the group velocity v_g = c .