1) Orbital angular momentum

Consider a particle described by the Cartesian coordinates $(x, y, z) \equiv \mathbf{r}$ and their conjugate momenta $(p_x, p_y, p_z) \equiv \mathbf{p}$. The classical definition of the orbital angular momentum of such a particle about the origin is $\mathbf{L} = \mathbf{r} \times \mathbf{p}$, giving

$$L_x = y\,p_z - z\,p_y, \tag{1}$$

$$L_y = z\,p_x - x\,p_z, \tag{2}$$

$$L_z = x\,p_y - y\,p_x. \tag{3}$$

Let us assume that the operators $(L_x, L_y, L_z) \equiv \mathbf{L}$ which represent the components of orbital angular momentum in quantum mechanics can be defined in an analogous manner to the corresponding components of classical angular momentum. In other words, we are going to assume that the above equations specify the angular momentum operators in terms of the position and linear momentum operators. Note that L_x, L_y, and L_z are Hermitian, so they represent things which can, in principle, be measured. Note, also, that there is no ambiguity regarding the order in which operators appear in products on the right-hand sides Eqs. (1)–(3), since all of the products consist of operators which commute.

The fundamental commutation relations satisfied by the position and linear momentum operators are

$$[x_i, x_j] = 0, \tag{4}$$

$$[p_i, p_j] = 0, \tag{5}$$

$$[x_i, p_j] = i\,\hbar\,\delta_{ij}, \tag{6}$$

where i and j stand for either x, y, or z. Consider the commutator of the operators L_x and L_z:

$$\begin{aligned} [L_x, L_y] &= [(y\,p_z - z\,p_y), (z\,p_x - x\,p_z)] = y\,[p_z, z]\,p_x + x\,p_y\,[z, p_z] \\ &= i\,\hbar\,(-y\,p_x + x\,p_y) = i\,\hbar\,L_z. \end{aligned}$$

The cyclic permutations of the above result yield the fundamental commutation relations satisfied by the components of an angular momentum:

$$[L_x, L_y] = i\,\hbar\,L_z, \tag{8}$$

$$[L_y, L_z] = i\,\hbar\,L_x, \tag{9}$$

$$[L_z, L_x] = i\,\hbar\,L_y. \tag{10}$$

These can be summed up more succinctly by writing

$$\mathbf{L} \times \mathbf{L} = i\,\hbar\,\mathbf{L}. \tag{11}$$

The three commutation relations (8)–(10) are the foundation for the whole theory of angular momentum in quantum mechanics. Whenever we encounter three operators having these commutation relations, we know that the dynamical variables which they represent have identical properties to those of the components of an angular momentum (which we are about to derive). In fact, we shall assume that *any three operators which satisfy the commutation relations (8)–(10) represent the components of an angular momentum.*

Suppose that there are N particles in the system, with angular momentum vectors $\mathbf{L}_i$ (where i runs from 1 to N). Each of these vectors satisfies Eq. (11), so that

$$\mathbf{L}_i \times \mathbf{L}_i = i\,\hbar\,\mathbf{L}_i. \tag{12}$$

However, we expect the angular momentum operators belonging to different particles to commute, since they represent different degrees of freedom of the system. So, we can write

$$\mathbf{L}_i \times \mathbf{L}_j + \mathbf{L}_j \times \mathbf{L}_i = 0, \tag{13}$$

for $i \neq j$. Consider the total angular momentum of the system, $\mathbf{L} = \sum_{i=1}^{N} \mathbf{L}_i$. It is clear from Eqs. (12) and (13) that

$$\begin{aligned}
\mathbf{L} \times \mathbf{L} &= \sum_{i=1}^{N} \mathbf{L}_i \times \sum_{j=1}^{N} \mathbf{L}_j = \sum_{i=1}^{N} \mathbf{L}_i \times \mathbf{L}_i + \frac{1}{2} \sum_{i,j=1}^{N} (\mathbf{L}_i \times \mathbf{L}_j + \mathbf{L}_j \times \mathbf{L}_i) \\
&= i\,\hbar \sum_{i=1}^{N} \mathbf{L}_i = i\,\hbar\,\mathbf{L}.
\end{aligned}$$

Consider the magnitude squared of the angular momentum vector, $L^2 \equiv L_x^{\,2} + L_y^{\,2} + L_z^{\,2}$. The commutator of L^2 and L_z is written

$$[L^2, L_z] = [L_x^{\,2}, L_z] + [L_y^{\,2}, L_z] + [L_z^{\,2}, L_z]. \tag{15}$$

It is easily demonstrated that

$$\begin{aligned} [L_x^{\,2}, L_z] &= -i\,\hbar\,(L_x\,L_y + L_y\,L_x), &(16)\\ [L_y^{\,2}, L_z] &= +i\,\hbar\,(L_x\,L_y + L_y\,L_x), &(17)\\ [L_z^{\,2}, L_z] &= 0, &(18) \end{aligned}$$

so

$$[L^2, L_z] = 0. \tag{19}$$

Since there is nothing special about the *z*-axis, we conclude that L^2 also commutes with L_x and L_y. It is clear from Eqs. (8)–(10) and (19) that the best we can do in quantum mechanics is to specify the magnitude of an angular momentum vector along with *one* of its components (by convention, the *z*-component).

It is convenient to define the *shift operators* L^+ and L^-:

$$\begin{aligned} L^+ &= L_x + i\,L_y, &(20)\\ L^- &= L_x - i\,L_y. &(21) \end{aligned}$$

Note that

$$[L^+, L_z] = -\hbar\,L^+, \tag{22}$$

$$\begin{aligned} [L^-, L_z] &= +\hbar\,L^-, &(23)\\ [L^+, L^-] &= 2\,\hbar\,L_z. &(24) \end{aligned}$$

Note, also, that both shift operators commute with L^2.

Eigenfunctions of orbital angular momentum

In Cartesian coordinates, the three components of orbital angular momentum can be written

$$
\begin{aligned}
L_x &= -i\hbar\left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}\right)\\
L_y &= -i\hbar\left(z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}\right) \qquad 26\\
L_z &= -i\hbar\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right)
\end{aligned}
$$

using the Schrödinger representation. Transforming to standard spherical polar coordinates,

$$x = r\sin\theta\cos\varphi, \tag{27}$$

$$y = r\sin\theta\sin\varphi, \tag{28}$$

$$z = r\cos\theta, \tag{29}$$

we obtain

$$L_x = i\hbar\left(\sin\varphi\frac{\partial}{\partial\theta} + \cot\theta\cos\varphi\frac{\partial}{\partial\varphi}\right) \tag{30}$$

$$L_y = -i\hbar\left(\cos\varphi\frac{\partial}{\partial\theta} - \cot\theta\sin\varphi\frac{\partial}{\partial\varphi}\right) \tag{31}$$

$$L_z = -i\hbar\frac{\partial}{\partial\varphi}. \tag{32}$$

$$L^2 = L_x{}^2 + L_y{}^2 + L_z{}^2 \tag{33}$$

from the above eqs. (30), (31) , (32) and (34), we can obtain:

$$L^2 = -\hbar^2\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\right). \tag{34}$$

The eigenvalue problem for L^2 takes the form

$$L^2\psi = \lambda\,\hbar^2\,\psi, \tag{35}$$

where $\psi(r,\theta,\varphi$ is the wave-function, and λ is a number. Let us write

$$(r,\theta,\varphi) = R(r)\,Y(\theta,\varphi). \tag{36}$$

Equation (35) reduces to

$$\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\right)Y + \lambda\,Y = 0, \tag{37}$$

where use has been made of Eq. (34) As is well-known, square integrable solutions to this equation only exist when λ takes the values $l\,(l+1)$, where l is an integer. These solutions are known as *spherical harmonics*, and can be written

$$Y_l^m(\theta,\varphi) = \sqrt{\frac{2\,l+1}{4\pi}\frac{(l-m)!}{(l+m)!}}\,(-1)^m\,e^{i\,m\,\varphi}\,P_l^m(\cos\varphi), \tag{38}$$

where m is a positive integer lying in the range $0 \leq m \leq l$. Here, $P_l^m(\xi)$ is an *associated Legendre function* satisfying the equation

$$\frac{d}{d\xi}\left[(1-\xi^2)\frac{dP_l^m}{d\xi}\right] - \frac{m^2}{1-\xi^2}P_l^m + l\,(l+1)\,P_l^m = 0. \tag{40}$$

We define

$$Y_l^{-m} = (-1)^m\,(Y_l^m)^*, \tag{41}$$

which allows m to take the negative values $-l \leq m < 0$. The spherical harmonics are *orthogonal* functions, and are properly normalized with respect to integration over the entire solid angle:

$$\int_0^\pi\int_0^{2\pi} Y_l^{m*}(\theta,\varphi)\,Y_{l'}^{m'}(\theta,\varphi)\,\sin\theta\,d\theta\,d\varphi = \delta_{ll'}\,\delta_{mm'}. \tag{42}$$

The spherical harmonics also form a complete set for representing general functions of θ and φ.

By definition,

$$L^2\, Y_l^m = l\,(l+1)\,\hbar^2\, Y_l^m, \tag{43}$$

where l is an integer. It follows from Eqs. (32)and (38) that

$$L_z\, Y_l^m = m\,\hbar\, Y_l^m, \tag{44}$$

where m is an integer lying in the range $-l \leq m \leq l$. Thus, the wave-function $(r, \theta, \varphi) = R(r)\, Y_l^m(\theta, \phi)$, where R is a general function, has all of the expected features of the wave-function of a simultaneous eigenstate of L^2 and L_z belonging to the quantum numbers l and m.