Chapter 1

Vector Algebra

1.1 Definitions

A **vector** consists of two components: *magnitude* and *direction* .
(`e.g.` force, velocity, pressure)

A **scalar** consists of *magnitude* only.
(`e.g.` mass, charge, density)

1.2 Vector Algebra

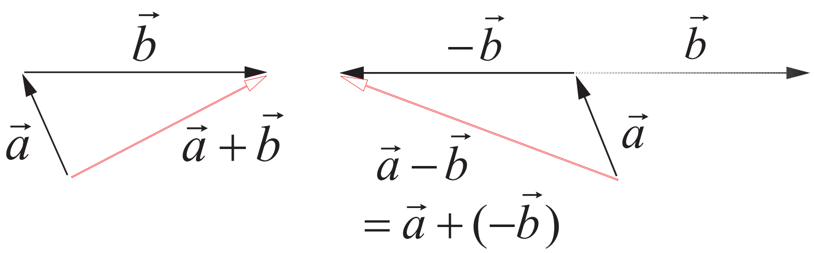


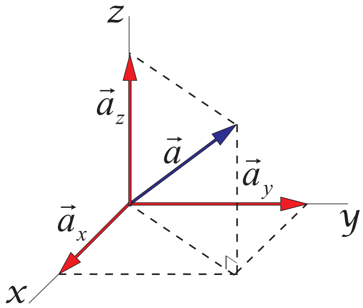
Figure 1.1: Vector algebra

$$
\begin{aligned}
\vec{a}+\vec{b} &= \vec{b}+\vec{a} \\
\vec{a}+(\vec{c}+\vec{d}) &= (\vec{a}+\vec{c})+\vec{d}
\end{aligned}
$$

1.3 Components of Vectors

Usually vectors are expressed according to **coordinate system**. Each vector can be expressed in terms of *components*.

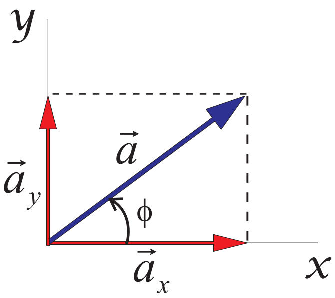
The most common coordinate system: **Cartesian**



$$\vec{a} = \vec{a}_x + \vec{a}_y + \vec{a}_z$$

Magnitude of $\vec{a} = |\vec{a}| = a$,

$$a = \sqrt{a_x^2 + a_y^2 + a_z^2}$$



$$\begin{aligned} \vec{a} &= \vec{a}_x + \vec{a}_y \\ a &= \sqrt{a_x^2 + a_y^2} \\ a_x &= a\,cos\phi;\; a_y \;=\; a\,sin\phi \\ tan\phi &= \frac{a_y}{a_x} \end{aligned}$$

Figure 1.2: ϕ measured anti-clockwise from position x-axis

Unit vectors have magnitude of 1

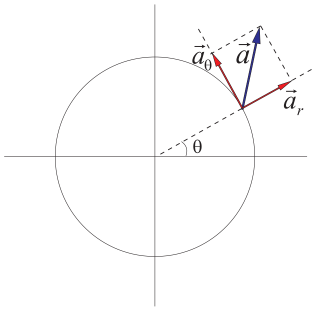
$$\hat{a} = \frac{\vec{a}}{|\vec{a}|} = \text{unit vector along } \vec{a} \text{ direction}$$

$$\begin{array}{cccl} \hat{i} & \hat{j} & \hat{k} & \text{are unit vectors along} \\ \updownarrow & \updownarrow & \updownarrow & \\ x & y & z & \text{directions} \end{array}$$

$$\vec{a} = a_x\,\hat{i} + a_y\,\hat{j} + a_z\,\hat{k}$$

Other coordinate systems:

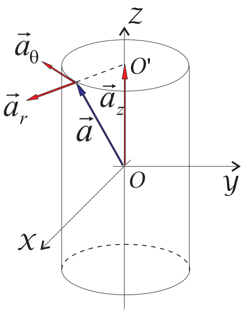
1. Polar Coordinate:



$$\vec{a} = a_r\,\hat{r} + a_\theta\,\hat{\theta}$$

Figure 1.3: Polar Coordinates

2. Cylindrical Coordinates:

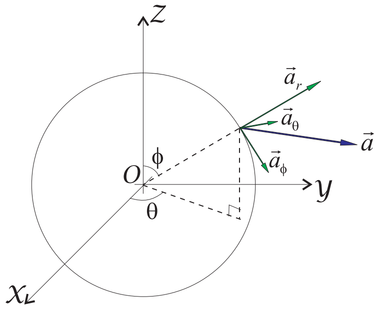


$$\vec{a} = a_r\,\hat{r} + a_\theta\,\hat{\theta} + a_z\,\hat{z}$$

$\hat{r}$ originated from nearest point on z-axis (Point O')

Figure 1.4: Cylindrical Coordinates

3. Spherical Coordinates:



$$\vec{a} = a_r\,\hat{r} + a_\theta\,\hat{\theta} + a_\phi\,\hat{\phi}$$

$\hat{r}$ originated from Origin O

Figure 1.5: Spherical Coordinates

1.4 Multiplication of Vectors

1. Scalar multiplication:

If	$\vec{b}$=m $\vec{a}$	$\vec{b}, \vec{a}$ are vectors; m is a scalar
then	b=m a	(Relation between magnitude)
	b_x=m a_x b_y=m a_y	$\}$ Components also follow relation

i.e.

$$\begin{array}{rcrcrcr} \vec{a} &=& a_x\,\hat{i} &+& a_y\,\hat{j} &+& a_z\,\hat{k} \\ m\vec{a} &=& ma_x\,\hat{i} &+& ma_y\,\hat{j} &+& ma_z\,\hat{k} \end{array}$$

2. Dot Product (Scalar Product):

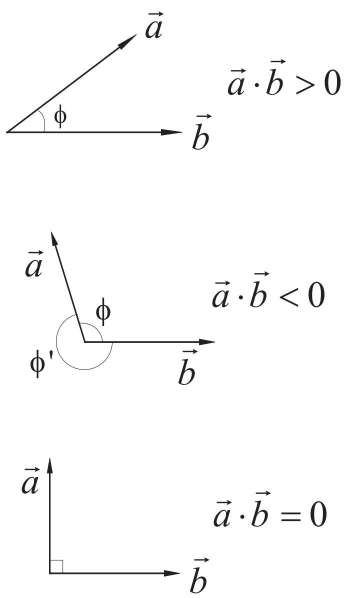


Figure 1.6: Dot Product

$$\boxed{\vec{a}\cdot\vec{b} = |\vec{a}|\cdot|\vec{b}|\;cos\phi}$$

Result is **always** a scalar. It can be positive or negative depending on ϕ.

$$\vec{a}\cdot\vec{b} = \vec{b}\cdot\vec{a}$$

Notice: $\vec{a}\cdot\vec{b} = ab\,cos\phi = ab\,cos\phi'$
i.e. Doesn't matter how you measure angle ϕ between vectors.

$$\begin{aligned} \hat{i}\cdot\hat{i} &= |\hat{i}|\,|\hat{i}|\;cos0^\circ\; = 1\cdot 1\cdot 1 = 1 \\ \hat{i}\cdot\hat{j} &= |\hat{i}|\,|\hat{j}|\,cos90^\circ = 1\cdot 1\cdot 0 = 0 \end{aligned}$$

$$\boxed{\begin{array}{c} \hat{i}\cdot\hat{i} = \hat{j}\cdot\hat{j} = \hat{k}\cdot\hat{k} = 1 \\ \hat{i}\cdot\hat{j} = \hat{j}\cdot\hat{k} = \hat{k}\cdot\hat{i} = 0 \end{array}}$$

If $\quad \vec{a} = a_x\,\hat{i} + a_y\,\hat{j} + a_z\,\hat{k}$

$\quad\;\; \vec{b} = b_x\,\hat{i} + b_y\,\hat{j} + b_z\,\hat{k}$

then $\quad \vec{a}\cdot\vec{b} = a_xb_x + a_yb_y + a_zb_z$

$\quad\;\; \vec{a}\cdot\vec{a} = |\vec{a}|\cdot|\vec{a}|\,cos0^\circ = a\cdot a = a^2$

3. Cross Product (Vector Product):

If $\quad \vec{c} = \vec{a} \times \vec{b}$,
then $\quad c = |\vec{c}| = a\, b\, sin\phi$

$$\vec{a} \times \vec{b} \neq \vec{b} \times \vec{a} \;\; \textbf{!!!}$$

$$\boxed{\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}}$$

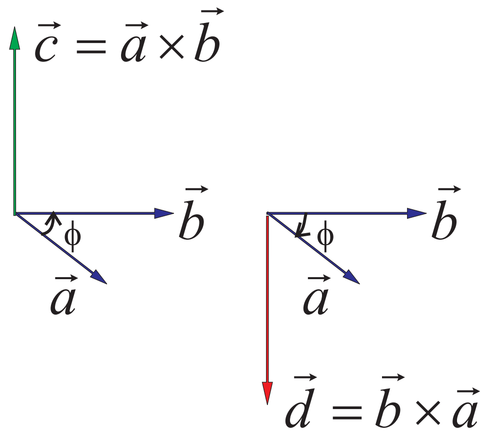


Figure 1.7: Note: How angle ϕ is measured

- Direction of cross product determined from *right hand rule*.
- Also, $\vec{a} \times \vec{b}$ is $\perp$ to $\vec{a}$ and $\vec{b}$, `i.e.`

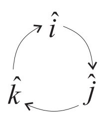
$$\vec{a} \cdot (\vec{a} \times \vec{b}) = 0$$
$$\vec{b} \cdot (\vec{a} \times \vec{b}) = 0$$

- IMPORTANT:

$$\boxed{\vec{a} \times \vec{a} = a \cdot a\, sin 0^\circ = 0}$$

$$\begin{aligned} |\hat{i} \times \hat{i}| &= |\hat{i}|\,|\hat{i}|\;\; sin0^\circ \;= 1 \cdot 1 \cdot 0 = 0 \\ |\hat{i} \times \hat{j}| &= |\hat{i}|\,|\hat{j}|\, sin90^\circ = 1 \cdot 1 \cdot 1 = 1 \end{aligned}$$

$$\boxed{\begin{array}{c} \hat{i} \times \hat{i} = \hat{j} \times \hat{j} = \hat{k} \times \hat{k} = 0 \\ \hat{i} \times \hat{j} = \hat{k};\; \hat{j} \times \hat{k} = \hat{i};\; \hat{k} \times \hat{i} = \hat{j} \end{array}}$$



$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix} = \begin{array}{r} (a_y\, b_z - a_z\, b_y)\,\hat{i} \\ +(a_z\, b_x - a_x\, b_z)\,\hat{j} \\ +(a_x\, b_y - a_y\, b_x)\,\hat{k} \end{array}$$

4. Vector identities:

$$\begin{aligned}\vec{a}\times(\vec{b}+\vec{c}) &= \vec{a}\times\vec{b}+\vec{a}\times\vec{c}\\ \vec{a}\cdot(\vec{b}\times\vec{c}) &= \vec{b}\cdot(\vec{c}\times\vec{a}) = \vec{c}\cdot(\vec{a}\times\vec{b})\\ \vec{a}\times(\vec{b}\times\vec{c}) &= (\vec{a}\cdot\vec{c})\,\vec{b}-(\vec{a}\cdot\vec{b})\,\vec{c}\end{aligned}$$

1.5 Vector Field (Physics Point of View)

A **vector field** $\vec{\mathcal{F}}(x,y,z)$ is a mathematical function which has a *vector* output for a *position* input.

(**Scalar field** $\vec{\mathcal{U}}(x,y,z)$)

1.6 Other Topics

Tangential Vector

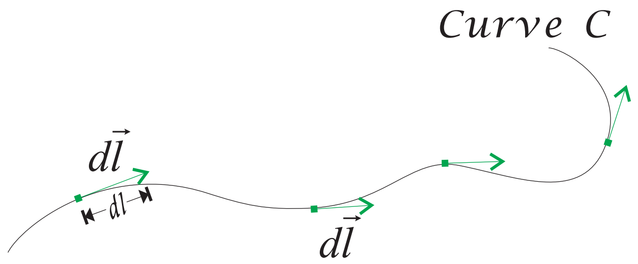


Figure 1.8: $d\vec{l}$ is a vector that is always tangential to the curve C with infinitesimal length dl

Surface Vector

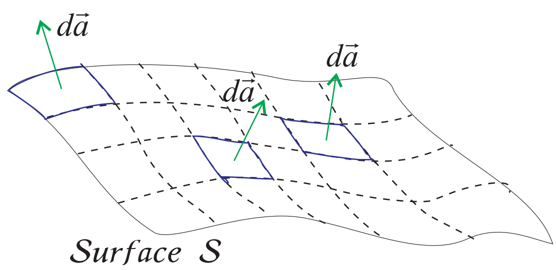


Figure 1.9: $d\vec{a}$ is a vector that is always perpendicular to the surface S with infinitesimal area da