# Error Analysis

## classification of control systems

control systems may be classified according to their ability to follow step inputs, ramp inputs, parabolic inputs, etc. This is a reasonable classification scheme because actual inputs may frequently be considered combinations of such inputs. Consider the following open-loop transfer function G(s) H(s):

$$G(s)H(s) = \frac{k(T_a s+1)(T_b s+1)\cdots(T_m s+1)}{s^N(T_1 s+1)(T_2 s+1)\cdots(T_p s+1)}$$

A system is called typ 0, typ 1, type 2, ... if $N=0$, $N=1$, $N=2$, ... respectively. If G(s) H(s) is written so that each term in the neumerator and denominator, except the term $s^N$, approaches unity as s approaches zero, then the open-loop gain k is directly related to the steady-state error.

## steady-state error

Consider the system shown in fig-1-. The closed-loop transfer function is

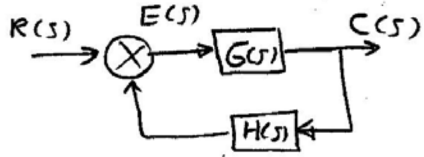


Fig-1-

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)H(s)}$$

$$\frac{E(s)}{R(s)} = 1 - \frac{C(s)H(s)}{R(s)} = \frac{1}{1+G(s)H(s)}$$

$$E(s) = \frac{1}{1+G(s)H(s)} R(s)$$

$$e_{ss} = \lim_{t\to\infty} e(t) = \lim_{s\to 0} \frac{s R(s)}{1+G(s)H(s)}$$

## Static position error coefficient $k_p$

The steady-state actuating error of the system for a unit-step input is

$$e_{ss} = \lim_{s \to 0} \frac{s}{1 + G(s)H(s)} \frac{1}{s} = \frac{1}{1 + G(0)H(0)}$$

The static position error coefficient $k_p$ is defined by

$$k_p = \lim_{s \to 0} G(s)H(s) = G(0)H(0)$$

$$e_{ss} = \frac{1}{1 + k_p}$$

For a type 0 system

$$k_p = \lim_{s \to 0} \frac{k(T_a s + 1)(T_b s + 1) \cdots}{(T_1 s + 1)(T_2 s + 1) \cdots} = k$$

For a type 1 or higher system

$$k_p = \lim_{s \to 0} \frac{k(T_a s + 1)(T_b s + 1) \cdots}{s^N (T_1 s + 1)(T_2 s + 1) \cdots} = \infty \quad (N \geq 1)$$

## Static velocity error coefficient $k_v$

The steady-state actuating error of the system with a unit-ramp input (unit-velocity input) is given by

$$e_{ss} = \lim_{s \to 0} \frac{s}{1 + G(s)H(s)} \frac{1}{s^2}$$

$$= \lim_{s \to 0} \frac{1}{s\,G(s)H(s)}$$

The static velocity error coefficient $k_v$ is defined by

$$k_v = \lim_{s \to 0} s\,G(s)H(s)$$

$$e_{ss} = \frac{1}{k_v}$$

The velocity error is not an error in velocity, but it is an error in position due to a ramp input.

For a type 0 system

$$k_v = \lim_{s \to 0} \frac{s\,k(T_a s+1)(T_b s+1)\cdots}{(T_1 s+1)(T_2 s+1)\cdots} = 0$$

For a type 1 system

$$k_v = \lim_{s \to 0} \frac{s\,k(T_a s+1)(T_b s+1)\cdots}{s(T_1 s+1)(T_2 s+1)\cdots} = k$$

For a type 2 or higher system,

$$k_v = \lim_{s \to 0} \frac{s\,k(T_a s+1)(T_b s+1)\cdots}{s^N(T_1 s+1)(T_2 s+1)\cdots} = \infty \quad (N \geq 2)$$

Static acceleration error coefficient $k_a$.

The steady-state actuating error of the system with a unit-parabolic input (acceleration input) which is defined by

$$r(t) = \frac{t^2}{2} \quad \text{for } t \geq 0$$
$$= 0 \quad \text{for } t < 0$$

is given by

$$e_{ss} = \lim_{s \to 0} \frac{s}{1+G(s)H(s)} \frac{1}{s^3}$$
$$= \frac{1}{\lim_{s \to 0} s^2 G(s)H(s)}$$

The static acceleration error coefficient $k_a$ is defined by the equation

$$k_a = \lim_{s \to 0} s^2 G(s)H(s)$$

$$e_{ss} = \frac{1}{k_a}$$

Note that the acceleration error, the steady-state error due to a parabolic input, is an error in position.

For a type 0 system

$$K_a = \lim_{s \to 0} \frac{s^2 K(T_a s+1)(T_b s+1)\cdots}{(T_1 s+1)(T_2 s+1)\cdots} = 0$$

For a type 1 system

$$K_a = \lim_{s \to 0} \frac{s^2 K(T_a s+1)(T_b s+1)\cdots}{s(T_1 s+1)(T_2 s+1)\cdots} = 0$$

For a type 2 system

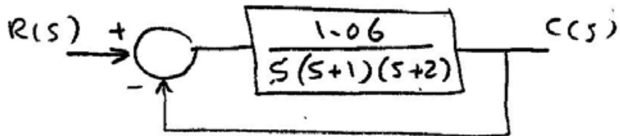
$$K_a = \lim_{s \to 0} \frac{s^2 K(T_a s+1)(T_b s+1)\cdots}{s^2(T_a s+1)(T_b s+1)\cdots} = K$$

For a type 3 or higher system

$$K_a = \lim_{s \to 0} \frac{s^2 K(T_a s+1)(T_b s+1)\cdots}{s^N(T_a s+1)(T_b s+1)\cdots} = \infty \quad (N \geq 3)$$

## Example

Consider the system shown below.




obtain the unit-step response and static velocity error

Sol.-

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)} = \frac{1.06}{s(s+1)(s+2)+1.06}$$

$$= \frac{1.06}{(s+2.33)(s+0.33+j0.58)(s+0.33-j0.58)}$$

- For unit step $R(s) = \frac{1}{s}$

$$c(t) = \mathcal{L}^{-1}[C(s)]$$

$$= 1 - 0.103e^{-2.33t} - e^{-0.33t}(0.897\cos 0.58t + 0.933\sin 0.58t)$$

$$K_v = \lim_{s \to 0} sG(s) = \frac{1.06}{2} = 0.53$$

$$e_{ss} = \frac{1}{K_v} = 1.89$$

**Problem 4·24.** *A unity feedback control system has a forward path transfer function given by*

$$G(s)=\frac{10}{s(s+1)}$$

*Find the value of the damping ratio and undamped natural frequency. If tachometer feedback is introduced so as to change the feedback path transfer function from unity to $(1+K_n s)$ what should be the value of $K_n$ to obtain damping ratio of 0·5. Find the percentage peak overshoot in each case.*

**Solution.** The characteristic equation is

$$1+G(s)=0$$

$$=s^2+s+10$$

$$\xi=\frac{1}{2\sqrt{10}}=0{\cdot}158$$

and

$$\omega_n=\sqrt{10} \text{ radians/sec}$$

$$=3{\cdot}162 \text{ radians/sec}$$

The closed loop system transfer function $=\frac{G}{1+GH}$ hence characteristic equation for closed loop system is

$$=1+GH=0$$

$$=1+\frac{10(1+K_n s)}{s(s+1)}=0$$

$$s^2+s+10+10K_n s=0$$

$$s^2+(1+10K_n)s+10=0$$

$$\xi=0{\cdot}5 \text{ (given)}$$

$$=\frac{1+10K_n}{2\sqrt{10}}$$

hence

$$3{\cdot}162=1+10K_n$$

$$K_n=0{\cdot}2162$$

Peak overshoot in first case

$$(\xi=0{\cdot}158)=63{\cdot}76\%$$

Peak overshoot when, $\xi=0{\cdot}5$ is 24%.

**Problem 4·25.** ***For unity feedback system having***

(1) $G(s)=\frac{K}{s(s+1)}$ ***and*** (2) $G(s)=\frac{K}{s^2(s+5)}$

***find*** (a) $K_p$, $K_v$, ***and*** $K_a$.

(b) ***Steady state error for unit step position, unit ramp and unit acceleration inputs.***

(c) ***Obtain generalized error coefficients and write the error series.***

**Solution.** Considering first

$$G(s)=\frac{K}{s(s+1)}$$

(a) **This is type 1 system hence,**

$$K_p=\infty\ ;\ K_v=K \text{ and } K_a=0$$

(b) and the steady state errors are for unit step position input

$$e_{ss}=\text{zero (since } e_{ss}=R/1+K_p)$$

For unit step velocity input $e_{ss}=1/K$ and

For unit step acceleration input $e_{ss}$=infinity.

(c) The generalised error coefficients are to be found from $w_e(s)$ which is,

$$\epsilon(s)=\frac{1}{1+G(s)}$$

hence, for the present case,

$$\epsilon(s)=\frac{1}{1+\frac{K}{s(s+1)}}$$

$$=\frac{s(s+1)}{s(s+1)+K}.$$

Now,

$$K_0=\lim_{s\to 0}\epsilon(s)=\text{zero}$$

$$K_1=\lim_{s\to 0}\frac{d[\epsilon(s)]}{ds}$$

$$=\lim_{s\to 0}\left[\frac{d}{ds}\left(\frac{s(s+1)}{s(s+1)+K}\right)\right]$$

$$=\frac{1}{K}$$

$$K_2 = \lim_{s\to 0} \frac{d^2[\epsilon(s)]}{ds^2}$$

$$= \lim_{s\to 0} \left\{ \frac{d^2}{ds^2} \left[ \frac{s(s+1)}{s(s+1)+K} \right] \right\}$$

$$= \frac{2(K-1)}{K^2}$$

So the error series is,

$$e_{ss}(t) = K_0 r(t) + K_1 \frac{dr(t)}{dt} + \frac{K_2}{2!} \frac{d^2 r(t)}{dt^2} + \ldots\ldots$$

$$= 0 + \frac{1}{K} \frac{dr(t)}{dt} + \frac{2(K-1)}{2K^2} \frac{d^2 r(t)}{dt^2} + \ldots\ldots$$

*Check.* Which shows :

(*i*) for unit step input *i.e.* $r(t)=u(t)$, $e_{ss}=$ zero

(*ii*) for unit ramp input *i.e.* $r(t)=t$,

$$e_{ss} = \frac{1}{K}$$

(*iii*) for unit acceleration input *i.e.* $r(t)=t^2$, $e_{ss}=$ infinity which are the results obtained in part (*b*).

Let $$G(s) = \frac{K}{s^2(s+5)(s+2)}$$

$$= \frac{K/10}{s^2\left(1+\frac{s}{5}\right)\left(1+\frac{s}{5}\right)}$$

This is type 2 system hence,

(*a*) $K_p=\infty$, $K_v=\infty$ and $K_a$ is

$$\lim_{s\to 0} s^2 G(s) = \lim_{s\to 0} \frac{s^2 K/10}{s^2\left(1+\frac{s}{5}\right)\left(1+\frac{s}{2}\right)}$$

$$= \frac{K}{10}$$

(*b*) The steady state errors are :

(*i*) For Unit step position input :

$$e_{ss} = \text{zero}$$

(*ii*) For Unit step-velocity input :

$$e_{ss} = \text{zero}$$

(*iii*) For Unit step-acceleration input :

$$e_{ss} = \frac{1}{K_a} = \frac{10}{K}$$

(*c*) For the generalized error coefficients

$$\epsilon(s) = \frac{1}{1+G(s)}$$

$$= \frac{1}{1 + \dfrac{K/10}{s^2\left(1+\dfrac{s}{5}\right)\left(1+\dfrac{s}{2}\right)}}$$

$$= \frac{s^2(s+5)(s+2)}{s^2(s+5)(s+2)+K}$$

Therefore,

$$K_0 = \lim_{s\to 0} \epsilon(s) = 0$$

$$K_1 = \lim_{s\to 0} \left[\frac{d}{ds}\ \epsilon(s)\right] = 0$$

$$K_2 = \lim_{s\to 0} \left[\frac{d^2}{ds^2}\ \epsilon(s)\right]$$

$$= \frac{10}{K}$$

Hence the error series are

$$e(t) = 0+0+\frac{10}{K}\ \frac{d^2 r(t)}{dt^2} + \text{higher order terms}$$

**Problem 4·26.** *Find the actuating signal $\epsilon$ for the system shown in the Fig. 4·40. Find position, velocity and acceleration constants and explain their significance in this general case.*

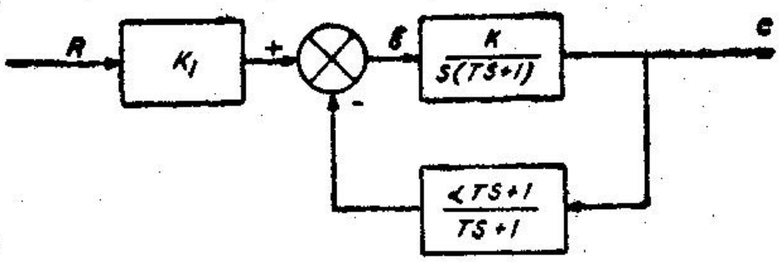



Fig. 4·39.

**Solution.** $\epsilon = RK_1 - C\dfrac{\alpha Ts+1}{Ts+1}$ and $\dfrac{\epsilon K}{s(Ts+1)} = C$

$$\therefore \qquad \epsilon = RK_1 - \left(\frac{\alpha Ts+1}{Ts+1}\right) \cdot \frac{\epsilon K}{s(1+Ts)}$$

*i.e.*
$$\epsilon\left[1 + \frac{\alpha Ts+1}{Ts+1} \cdot \frac{K}{s(1+Ts)}\right] = RK_1$$

*i.e.*
$$\epsilon = K_1 R(s) \Bigg/ \left[1 + \frac{(\alpha Ts+1)K}{s(1+Ts)^2}\right]$$

For step input $R(s) = R/s$

$$\therefore \qquad \epsilon(s) = \frac{K_1\ R/s}{\left[1 + \dfrac{(\alpha Ts+1)\ K}{s(1+Ts)^2}\right]}$$

$$\epsilon_{ss} = \lim_{s\to 0} s\epsilon(s) = \lim_{s\to 0}\left[\frac{K_1 R}{1 + \dfrac{(\alpha Ts+1)\ K}{s(1+Ts)^2}}\right]$$

$$\therefore \qquad \epsilon_{ss} = 0 = \frac{R}{1+K_p}$$

where $\qquad K_p$ = position error constant.

So $\qquad K_p = \infty$ ($\because \quad R \neq 0$)

for velocity input $R(s) = \dfrac{R_1}{s^2}$

$$\epsilon_{ss} = \lim_{s\to 0}\frac{R_1 K_1/s}{\left[1 + \dfrac{(\alpha Ts+1)\ K}{s(1+Ts)^2}\right]} = \lim_{s\to 0}\frac{K_1 R_1\ (1+Ts)^2}{s(1+Ts)^2 + (\alpha Ts+1)K}$$

$$\epsilon_{ss} = \frac{K_1 R_1}{K} = \frac{R_1}{K_v}$$

where $K_v$ is velocity error constant.

$$\therefore \qquad K_v = K/K_1$$

Similarly for acceleration input

$$R(s) = R_2{}^2/s^3$$

and
$$\underset{s\to\infty}{\epsilon_{ss}} = \frac{R^2}{K_a}$$

$$\therefore \qquad K_a = 0$$

since $\qquad R_2 \neq \infty$

$\therefore$ For this system

$$K_p = \infty, \quad K_v = \frac{K}{K_1}$$

and $\qquad K_a = 0$

**Problem 4·27.** *For the system of the Fig. 4·40 find the steady state error for step velocity, position and acceleration input.*

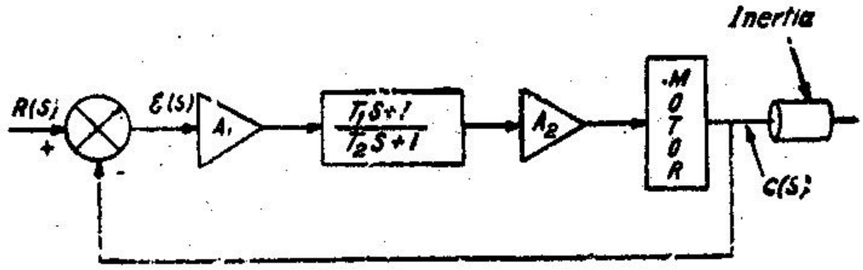



Fig. 4·40.

**Solution** $T_m = KV_1(s) + msc(s)$

$T_m = Js^2\theta_1 + fs\theta_1 = Js^2c + fsc$

$Js^2\theta_1 + fs\theta_1 = KV_1(s) + msc$

Now $V_1 = \epsilon A_1 A_2 \dfrac{T_1s+1}{T_2s+1}$ and $R - c = \epsilon$

$$[Js^2c + fsc - msc) = \epsilon\ KA_1A_2\frac{T_1s+1}{T_2s+1}$$

or

$$[Js^2 + (f-m)s][(R-\epsilon)] = \epsilon KA_1A_2\frac{T_1s+1}{T_2s+1}$$

$$\epsilon\left[KA_1A_2\frac{T_1s+1}{T_2s+1} + Js^2 + (f-m)s\right] = [Js^2 + (f-m)s]R$$

$$\epsilon(s) = \frac{[Js^2 + (f-m)s](T_2s+1)\ R(s)}{[KA_1A_2(T_1s+1)\{Js^2 + (f-m)s\}(T_2s+1)]}$$

For step position input

$$R(s) = \frac{R}{s}$$

$$\epsilon_{ss} = \lim_{s\to 0} s\epsilon(s) = \lim_{s\to 0}\frac{[Js^2 + (f-m)s](T_2s+1)R}{[KA_1A_2(T_1s+1) + \{Js^2 + (f-m)s\}(T_2s+1)]} = 0$$

For velocity input

$$R(s) = R_1/s^2$$

$$\epsilon_{ss} = \lim_{s\to 0}\frac{[Js + (f-m)][T_2s+1]R_1}{[KA_1A_2(T_1s+1) + (Js + (f-m))][T_2s+1]}$$

$$= (f-m)R_1/KA_1A_2$$

for acceleration input

$$R(s) = \frac{R_1}{2\ s^3}$$

and $\therefore$ $\epsilon_{ss} = \infty$

**Problem 4·29.** *Find out the position, velocity and acceleration error coefficients for the unity feedback control system whose open loop transfer function is*

$$G(s)=\frac{50}{(1+0\cdot 1s)(1+2s)}.$$

**Solution.**

$$K_p=\lim_{s\to 0} G(s)$$

$$=\frac{50}{1\times 1}=50$$

$$K_v=\lim_{s\to 0} s^2G(s)$$

$$=\lim_{s\to 0}\frac{50\times s}{(1+0\cdot 1s)(1+2s)}=0$$

$$K_a=\lim_{s\to 0} s^2G(s)$$

$$=\lim_{s\to 0}\frac{50\times s^2}{(1+0\cdot 1s)(1+2s)}=0.$$

**Problem 4·30.** *Repeat problem 4·29 if*

$$G(s)=\frac{K}{s(1+0\cdot 1s)(1+0\cdot 5s)}.$$

**Solution.**

$$K_p=\lim_{s\to 0} G(s)$$

$$=\lim_{s\to 0} \frac{K}{s(1+0\cdot 1s)(1+0\cdot 5s)}=\infty$$

$$K_v=\lim_{s\to 0} sG(s)=\lim_{s\to 0} \frac{sK}{s(1+0\cdot 1s)\ (1+0\cdot 5s)}=K$$

$$K_a=\lim_{s\to 0} s^2G\ (s)=\lim_{s\to 0} \frac{s^2K}{s(1+0\cdot 1s)\ (1+0\cdot 5s)}=0$$

**Problem 4·31.** *Repeat problem 4·29 if*

$$G\ (s)=\frac{K}{s^2\ (s^2+4s+200)}$$

**Solution**

$$K_p=\lim_{s\to 0} G(s)=\lim_{s\to 0} \frac{K}{s\ (s^2+4s+200)}=\infty$$

$$K_v=\lim_{S\to 0} sG(s)=\infty$$

$$K_a=\lim_{s\to 0} s^2\ G(s)=\frac{s^2\ K}{s^2\ (s^2+4s\times 200)}=K/200$$

**Problem 4·32.** *Repeat problem 4·29 if*

$$G(s)=\frac{K\ (1+2s)\ (+4s)}{s^2\ (s^2+2s+10)}$$

**Solution.** $K_p=\lim_{s\to 0} G(s)=\infty$, $K_v=\lim_{s\to 0} s\ G(s)$

$\therefore$

$$K_v=\lim_{s\to 0} \frac{sK\ (1+2s)(1+4s)}{s^2\ (s^2+2s+10)}=\infty$$

and

$$K_a=\lim_{s\to 0} s^2G(s)=\lim_{s\to 0} \frac{s^2K\ (1+2s)\ (1+4s)}{s^2\ (s^2+2s+10)}$$

$$=K/10$$