EDEXCEL ANALYTICAL METHODS FOR ENGINEERS H1

UNIT 2 - NQF LEVEL 4

OUTCOME 1

TUTORIAL 1 – ALGEBRAIC METHODS

Algebraic methods: polynomial division; quotients and remainders; use of factor and remainder theorem; rules of order for partial fractions (including linear, repeated and quadratic factors); reduction of algebraic fractions to partial fractions

Exponential, trigonometric and hyperbolic functions: the nature of algebraic functions; relationship between exponential and logarithmic functions; reduction of exponential laws to linear form; solution of equations involving exponential and logarithmic expressions; relationship between trigonometric and hyperbolic identities; solution of equations involving hyperbolic functions

Arithmetic and geometric: notation for sequences; arithmetic and geometric progressions;
the limit of a sequence; sigma notation; the sum of a series; arithmetic and geometric series; Pascal's triangle and the binomial theorem

Power series: expressing variables as power series functions and use series to find approximate values (eg exponential series, Maclaurin's series, binomial series)

You should judge your progress by completing the self assessment exercises.

This is a useful web site that explains partial fractions
http://library.thinkquest.org/C0110248/algebra/pfintro.htm

1. PARTIAL FRACTIONS

Partial fractions are a useful way of simplifying polynomial expressions. This is done so that they are easier to integrate or expand. The degree of factor refers to the highest power of x in any factor (terms inside the brackets).

FIRST DEGREE LINEAR FACTORS

In the following expression, we have factors containing x^1 and x^0 so they are linear and first order. (Remember x^0 is 1 and not normally written and x^1 is normally written as x).

$$\frac{f(x)}{(x-a)(x-b)(x-c)}$$

Every time two linear factors are multiplied the powers increase by 1 so if we have three factors on the bottom line, the denominator has a highest power of 3. So long as the numerator f(x) has a lower power then the expression is exactly the same as the sum of three partial fractions such that:

$$\frac{f(x)}{(x-a)(x-b)(x-c)} = \frac{A}{x-a} + \frac{B}{x-b} + \frac{C}{x-c}$$

The method of solution is shown in the following examples.

WORKED EXAMPLE No.1

Find the partial fractions for $\frac{3x+2}{(x-2)(x+5)}$

SOLUTION

$$\frac{3x+2}{(x-2)(x+5)} \equiv \frac{A}{(x-2)} + \frac{B}{(x+5)}$$

Multiply through by (x – 2) $\quad \frac{3x+2}{(x+5)} \equiv A + \frac{B(x-2)}{(x+5)}$

The relationship must be valid for any value of x so conveniently put x = 2

$$\frac{6+2}{7} \equiv A + \frac{B(0)}{7} \quad \text{Hence } A = 8/7$$

Now start again and this time multiply through by (x + 5) and make x = -5

$$\frac{3x+2}{(x-2)} \equiv \frac{A(x+5)}{(x-2)} + B \qquad \frac{-13}{-7} \equiv \frac{A(0)}{(x-2)} + B \quad \text{Hence } B = 13/7$$

In the last example we could use the following method to obtain the result. To find A cover the factor (x – 2) and put x = 2 in the remainder.

$$\frac{3x+2}{(x-2)(x+5)} \qquad A = \frac{3(2)+2}{(2+5)} = \frac{8}{7}$$

To find B cover the factor (x +5) and put x = -5 in the remainder.

$$\frac{3x+2}{(x-2)(x+5)} \qquad B = \frac{3(-5)+2}{(-5-2)} = \frac{13}{7}$$

When we have three polynomials on the bottom line we need three partial fractions but we can still use the cover up method.

WORKED EXAMPLE No.2

Find the partial fractions for $\frac{x^2+2x+6}{(x-1)(x+3)(x-5)}$

SOLUTION

$$\frac{x^2+2x+6}{(x-1)(x+3)(x-5)} \equiv \frac{A}{(x-1)} + \frac{B}{(x+3)} + \frac{C}{(x-5)}$$

Cover (x – 1) and put x = 1 $\quad A = \frac{(1)^2+2(1)+6}{(1+3)(1-5)} = -\frac{9}{16}$

Cover (x + 3) put x = -3 $\quad B = \frac{9-6+6}{(-4)(-8)} = \frac{9}{32}$

Cover (x – 5) put x = 5 $\quad C = \frac{25+10+6}{(4)(8)} = \frac{41}{32}$

hence $\frac{x^2+2x+6}{(x-1)(x+3)(x-5)} \equiv -\frac{9}{16(x-1)} + \frac{9}{32(x+3)} + \frac{41}{32(x-5)}$

When the numerator has a higher degree than the denominator, we can use long division to reduce the expression until the remainder meets this condition.

REMAINDER THEOREM

When we divide a number by another and the result is not exact, it leaves a remainder. For example:

$\frac{15}{6} = 2 + \text{remainder } 3$ or in other words $2 + \frac{3}{6} = 2.5$

In this case 6 is the divisor, 2 is the quotient and 3 is the remainder.
We can say that 15 = 6 x 2 + 3

In the same way when we divide polynomials we can say:

Divisor × quotient + remainder ≡ polynomial

In particular if the divisor is (x + a) and the polynomial is f (x) then

f (x) ≡ (x + a) × quotient + remainder.

If x = -a then f (a) = (a – a) × quotient + remainder.

f (a) = remainder

This gives an easy way of finding the remainder when a polynomial is divided by (x – a)

THE FACTOR THEOREM

If the remainder is zero, then the polynomial will factorise.

WORKED EXAMPLE No.3

Divide $\dfrac{3x^3 + 4x^2 - 5x + 3}{(x+2)}$ and find the remainder by the remainder theorem.

SOLUTION

The first term of the resulting polynomial must be $3x^3 \div x = 3x^2$
Multiply $(x + 2)(3x^2) = 3x^3 + 6x^2$
The reminder is found by subtracting.

$$\begin{array}{l} 3x^3 + 4x^2 - 5x + 3 \\ \underline{3x^3 + 6x^2} \\ \quad\; -2x^2 - 5x + 3 \end{array}$$

Now divide again $(-2x^2 - 5x + 3) \div (x + 2)$
The result must be $(-2x^2) \div x = -2x$
$(x + 2)(-2x) = -2x^2 - 4x$
The reminder is found by subtracting.

$$\begin{array}{l} -2x^2 - 5x + 3 \\ \underline{-2x^2 - 4x^2} \\ \quad\; - x + 3 \end{array}$$

Now divide again $(-x + 3) \div (x + 2)$
The result must be $(-x) \div x = -1$
$(x + 2)(-1) = -x - 2$
The reminder is found by subtracting.

$$\begin{array}{l} -x + 3 \\ \underline{-x - 2} \\ \quad\; 5 \end{array}$$

The whole process is usually shown like this

$$\begin{array}{r|l} & 3x^2 - 2x - 1 \\ \hline x + 2 & 3x^3 + 4x^2 - 5x + 3 \\ & \underline{3x^3 + 6x^2} \\ & \quad -2x^2 - 5x \\ & \quad \underline{-2x^2 - 4x} \\ & \qquad\quad -x + 3 \\ & \qquad\quad \underline{-x - 2} \\ & \qquad\qquad 5 \end{array}$$

The result of the division is the polynomial $3x^2 - 2x - 1$ and the remainder is 5

The final remainder is 5. This may be found using the remainder theorem as follows.
The divisor is $x + 2$ use $x = -2$

$$f(-2) = (-2+2)(3\times(-2)^2 - 2\times(-2) - 1) + 5$$
$$= 0\times(3\times(-2)^2 - 2\times(-2) - 1) + 5 = 5$$

WORKED EXAMPLE No.4

Find the partial fractions for the expression $\frac{x^3+x+6}{(x-2)(x-4)}$

SOLUTION

Multiply out the brackets $\frac{x^3+x+6}{(x-2)(x-4)}=\frac{x^3+x+6}{x^2-6x+8}$

If we now divide out using long division we get:

$$x^2-6x+8\overline{\big)x^3+x+6} = x \text{ remainder } \left(x^3+x+6\right) - \left(x^3-6x^2+8x\right) = 6x^2-7x+6$$

$$x^2-6x+8\overline{\big)6x^2-7x+6} = 6 \text{ remainder } \left(6x^2-7x+6\right) - \left(6x^2-36x+48\right) = 29x-42$$

hence $\frac{x^3+x+6}{x^2-6x+8}=x+6+\frac{29x-42}{x^2-6x+8}$

The remainder now has a lower degree than the denominator so we can change this to partial fractions.

$x^2-6x+8=(x-4)(x-2)$ $\frac{29x-42}{x^2-6x+8}=\frac{29x-42}{(x-4)(x-2)}$ $\frac{29x-42}{(x-4)(x-2)}\equiv\frac{A}{x-4}+\frac{B}{x-2}$

$A=\frac{29(4)-42}{(4-2)}=37$ $B=\frac{29(2)-42}{(2-4)}=-8$

hence $\frac{x^3+x+6}{(x-2)(x-4)}=x+6+\frac{37}{x-4}-\frac{8}{x-2}$

REPEATED FACTORS

If we have two or more identical factors, they can be written as the factor to a suitable power. This is called a repeated factor. I can be shown that the partial fractions take the following form.

$$\frac{1}{(x-a)(x-b)^3} \equiv \frac{A}{x-a} + \frac{B}{x-b} + \frac{C}{(x-b)^2} + \frac{D}{(x-b)^3}$$

The method of solution is shown in the next example.

WORKED EXAMPLE No.5

Find the partial fractions for the expression $\frac{1}{(x-2)(x+1)^3}$

SOLUTION

The partial factors take the form of : $\frac{1}{(x-2)(x+1)^3} \equiv \frac{A}{x-2} + \frac{B}{x+1} + \frac{C}{(x+1)^2} + \frac{D}{(x+1)^3}$

A can be solved by the cover up method. However, when we multiply both sides by $(x+1)^3$ and put x = -1, only D will remain so D may be solved by the cover up method.

$$A = \frac{1}{(2+1)^3} = \frac{1}{27} \qquad D = \frac{1}{(-1-2)} = -\frac{1}{3}$$

We now have: $\frac{1}{(x-2)(x+1)^3} \equiv \frac{1}{27(x-2)} + \frac{B}{x+1} + \frac{C}{(x+1)^2} - \frac{1}{3(x+1)^3}$

Multiply both sides by $(x-2)(x+1)^3$ and:

$$1 \equiv \frac{(x-2)(x+1)^3}{27(x-2)} + \frac{B(x-2)(x+1)^3}{x+1} + \frac{C(x-2)(x+1)^3}{(x+1)^2} - \frac{(x-2)(x+1)^3}{3(x+1)^3}$$

$$\frac{(x+1)^3}{27} + B(x-2)(x+1)^2 + C(x-2)(x+1) - \frac{(x-2)}{3} - 1 = 0$$

Multiply out in full $\frac{x^3}{27} + \frac{3x^2}{27} + \frac{3x}{27} + \frac{1}{27} + Bx^3 - 3Bx - 2B + Cx^2 - Cx - 2C - \frac{(x-2)}{3} - 1 = 0$

The coefficients of x^3 must add up to zero so 1/27 + B = 0   B = -1/27

It is clear that we did not need to multiply out to obtain this result as the coefficients of x^3 are clear without doing this.

To find C we can equate the coefficients of x^2 and it is clear that 3/27 + C = 0 and C = -3/27 but this is not so obvious without multiplying out.

Another way is to equate coefficients of x^o (i.e. the constants)

$$\frac{1}{27} - 2B - 2C + \frac{2}{3} - 1 = 0 \qquad 2C = \frac{1}{27} - 2\left(-\frac{1}{27}\right) + \frac{2}{3} - 1 = \frac{1}{27} + \frac{2}{27} + \frac{18}{27} - \frac{27}{27} = -\frac{6}{27}$$

C = - 3/27

QUADRATIC (SECOND DEGREE) FACTORS

Second degree factors are brackets that contain x^2 in them. These should be factorised if possible to change them into first degree factors. If they are not easily factorised it can be shown that for each quadratic factor there is a partial fraction in the following form.

$$\frac{f(x)}{x^2+ax+b} \equiv \frac{Cx+D}{(x^2+ax+c)}$$

A repeated quadratic factor is dealt with in the same way as before hence:

$$\frac{f(x)}{(x^2+ax+b)^2} \equiv \frac{Cx+D}{(x^2+ax+c)} + \frac{Ex+F}{(x^2+ax+c)^2}$$

The method of solution is outlined in the following example.

WORKED EXAMPLE No.6

Find the partial fractions for the expression $\frac{x^2+3x+2}{(x-1)(x-2)(x^2+x+5)}$

SOLUTION

The partial factors take the form of : $\frac{A}{(x-1)} + \frac{B}{(x-2)} + \frac{Cx+D}{(x^2+x+5)}$

A and B are found by the cover up method.

$$A = \frac{1^2+3(1)+2}{(1-2)(1^2+1+5)} = -\frac{6}{7} \qquad B = \frac{2^2+3(2)+2}{(2-1)(2^2+2+5)} = \frac{12}{11}$$

$$\frac{x^2+3x+2}{(x-1)(x-2)(x^2+x+5)} \equiv -\frac{6}{7(x-1)} + \frac{12}{11(x-2)} + \frac{Cx+D}{(x^2+x+5)}$$

$$x^2+3x+2 \equiv -\frac{6(x-2)(x^2+x+5)}{7} + \frac{12(x-1)(x^2+x+5)}{11} + (Cx+D)(x-1)(x-2)$$

Equate coefficients of x^3

$$0 = -\frac{6}{7} + \frac{12}{11} + C \qquad C = \frac{66-84}{77} = -\frac{18}{77}$$ (multiply it out in full if you need convincing)

Equate coefficients of x^o (i.e. the constant terms)

$$2 \equiv \frac{6(10)}{7} - \frac{12(5)}{11} + 2D \qquad 2D = 2 - \frac{60}{7} + \frac{60}{11} = 2 - \frac{660}{77} + \frac{420}{77} = 2 - \frac{240}{77}$$

$$D = 1 - \frac{120}{77} = -\frac{43}{77}$$

$$\frac{x^2+3x+2}{(x-1)(x-2)(x^2+x+5)} \equiv -\frac{6}{7(x-1)} + \frac{12}{11(x-2)} + \frac{-(18/77)x - 43/77}{(x^2+x+5)}$$

$$\frac{x^2+3x+2}{(x-1)(x-2)(x^2+x+5)} \equiv -\frac{6}{7(x-1)} + \frac{12}{11(x-2)} - \frac{18x+43}{77(x^2+x+5)}$$

SERIES EXPANSION

The following example shows how partial fractions help us to expand an expression.

WORKED EXAMPLE No.7

Simplify the following expression so that it is a series in ascending orders of x.

$$\frac{1}{(2x+3)(x^2+1)}$$

SOLUTION

$$\frac{1}{(2x+3)(x^2+1)} \equiv \frac{A}{2x+3} + \frac{Bx}{x^2+1}$$

Use the cover up method to find A putting x = -1.5 $\qquad A = \frac{1}{(-1.5)^2+1} = \frac{4}{13}$

$$1 \equiv \frac{4(2x+3)(x^2+1)}{13(2x+3)} + \frac{(Bx+C)(2x+3)(x^2+1)}{x^2+1}$$

$$1 \equiv \frac{4(x^2+1)}{13} + (Bx+C)(2x+3)$$

Equate the coefficients of x^2 $\qquad 0 \equiv \frac{4}{13} + 2B \quad B = -\frac{2}{13}$

Equate the constants. $1 \equiv \frac{4}{13} + 3C \quad C = \frac{3}{13}$

$$\frac{1}{(2x+3)(x^2+1)} \equiv \frac{4}{13(2x+3)} + \frac{\left(-\frac{2}{13}\right)x + \frac{3}{13}}{x^2+1} \equiv \frac{1}{13}\left[\frac{4}{(2x+3)} - \frac{2x-3}{(x^2+1)}\right]$$

Rearrange into the required form.

$$\frac{1}{(2x+3)(x^2+1)} \equiv \frac{4}{13(3+2x)} - \frac{2x-3}{13(1+x^2)} \equiv \left[\frac{4}{39\left(1+\frac{2x}{3}\right)} - \frac{1}{13}(2x-3)(1+x^2)^{-1}\right]$$

$$\frac{1}{(2x+3)(x^2+1)} \equiv \frac{4}{39}\left(1+\frac{2x}{3}\right)^{-1} - \frac{1}{13}(2x-3)(1+x^2)^{-1}$$

Note this can now be expanded to form: $\qquad \frac{1}{(2x+3)(x^2+1)} \equiv \frac{13}{39} - \frac{26}{117}x - \frac{65}{351}x^2 + ...$

SELF ASSESSMENT EXERCISE No.1

Find the partial fractions for the following expressions.

1. $\dfrac{2x+5}{(x+3)(x+2)}$

2. $\dfrac{4x-17}{(x-2)(x-5)}$

3. $\dfrac{x^2+2x+5}{(x^2+2x-2)(x+1)}$

4. $\dfrac{x^2-2x+3}{(x^2-x-1)(x-1)}$

5. $\dfrac{4-x}{(x-3)^2}$

6. $\dfrac{(x+2)^2}{(x-3)^2(x+5)}$

7. $\dfrac{(x-3)}{(x-1)^2(x^2+2)}$

Answers

1. $\dfrac{1}{(x+3)}+\dfrac{1}{(x+2)}$

2. $\dfrac{3}{(x-2)}+\dfrac{1}{(x-5)}$

3. $\dfrac{7(x+1)}{3(x^2+2x-2)}-\dfrac{4}{3(x+1)}$

4. $\dfrac{(3x-1)}{(x^2-x-1)}-\dfrac{2}{(x-1)}$

5. $\dfrac{1}{(x-3)^2}-\dfrac{1}{(x-3)}$

6. $\dfrac{25}{8(x-3)^2}+\dfrac{55}{64(x-3)}+\dfrac{9}{64(x+5)}$

7. $\dfrac{7}{9(x-1)}-\dfrac{2}{3(x-1)^2}-\dfrac{(1+7x)}{9(x^2+2)}$