MATHEMATICS FOR ENGINEERING

TRIGONOMETRY

TUTORIAL 3 – PERIODIC FUNCTIONS

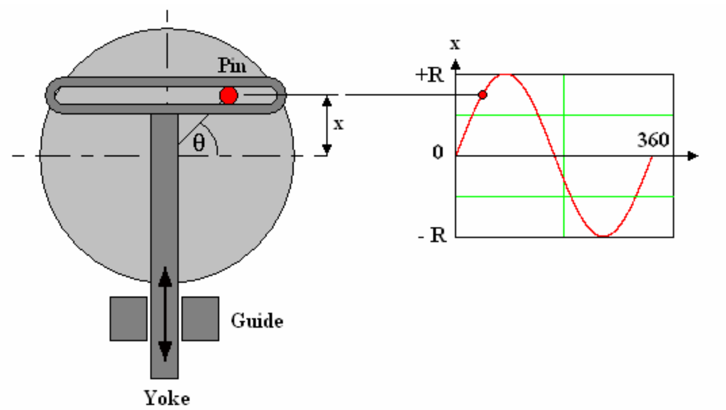
This is the one of a series of basic tutorials in mathematics aimed at beginners or anyone wanting to refresh themselves on fundamentals. The tutorial revises and extends the work on periodic motion and contains the following.

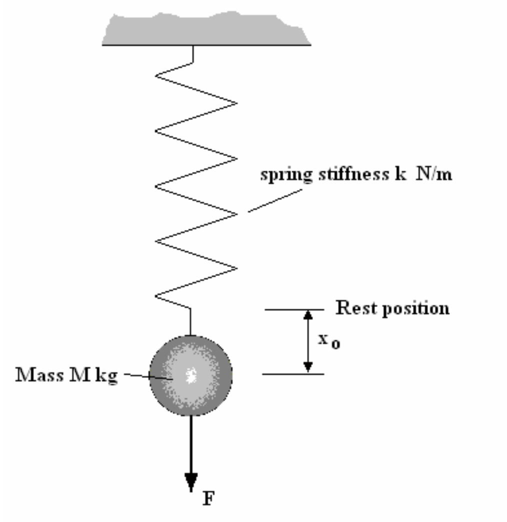
- Periodic Motion
  Graphical representations
  Angular frequency
  Periodic time
- Phase and Displacement
- Square waves
- Phasor diagrams
- Complex numbers

INTRODUCTION

In Nature and in Engineering there are many things that oscillate in some form or other and produce a repetitive change of some quantity with respect to time. Examples are mechanical oscillations and alternating electricity. In many cases a plot of the quantity against time produces a sinusoidal graph and the change is said to be sinusoidal.

Here are some examples.





SCOTCH YOKE MASS ON A SPRING

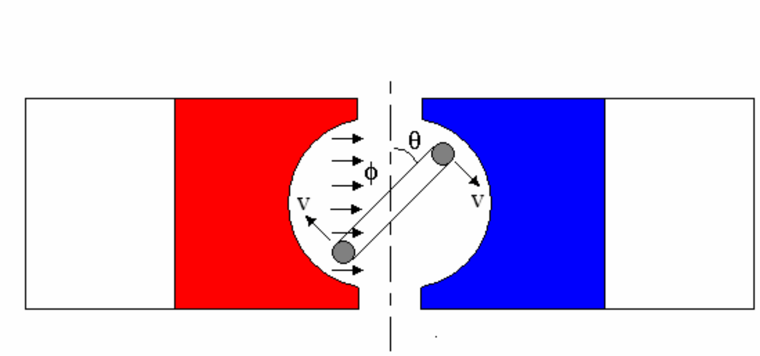
The Scotch Yoke is a device that produces up and down motion when the wheel is rotated. The displacement of the yoke from the horizontal position is $x = R \sin \theta$. Plotting x against time or angle will produce a sinusoidal graph.

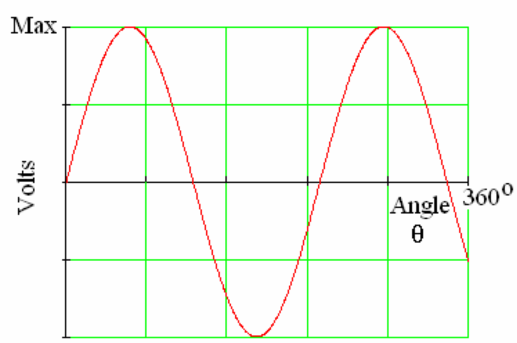
If the angular velocity of the wheel is ω rad/s then the angle is $\theta = \omega t$ radian.

A mass on the end of a spring will oscillate up and down and produce identical motion to the Scotch Yoke without the rotation of a wheel. The displacement is $x = A \sin (\omega t)$

ALTERNATING ELECTRICITY

Electricity is generated by rotating a conductor relative to a magnetic field at angular velocity ω rad/s (very simplified case shown). The voltage generated is directly proportional to the angle of rotation.





This explains why the voltage in our mains electrical system is sinusoidal. The voltage at any moment in time is given by the equation $v = V \sin(\omega t)$ where V is the maximum voltage (amplitude) in the cycle and ω the angular velocity or frequency.

ANGULAR FREQUENCY, FREQUENCY AND PERIODIC TIME

ω may be a physical angular velocity (e.g. the Scotch Yoke and the simple Generator) but in many cases it is simply referred to as the angular frequency as in the case of the mass on a spring.

The frequency of the wheel in revolutions/second is equivalent to the frequency of the oscillation. If the wheel rotates at 2 rev/s the time of one revolution is 1/2 seconds. If the wheel rotates at 5 rev/s the time of one revolution is 1/5 second. If it rotates at f rev/s the time of one revolution is 1/f. This formula is important and gives the periodic time.

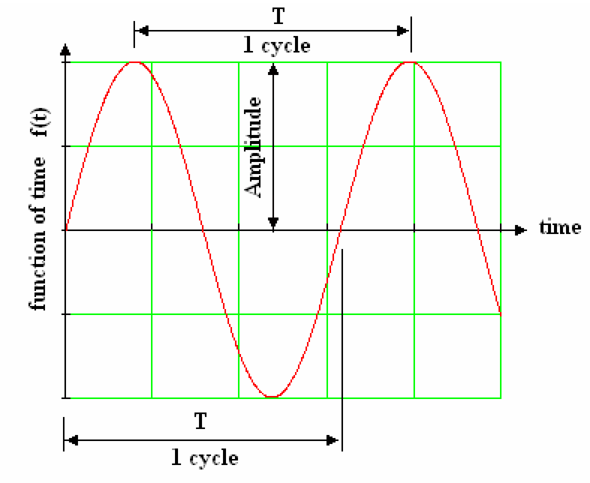
Periodic Time T = time needed to perform one cycle.

f is the frequency or number of cycles per second or Hertz (Hz).

It follows that **$T = 1/f$ and $f = 1/T$**

Each cycle of an oscillation is equivalent to one rotation of the wheel and 1 revolution is an angle of 2π radians. When $\theta = 2\pi$, $t = T$.

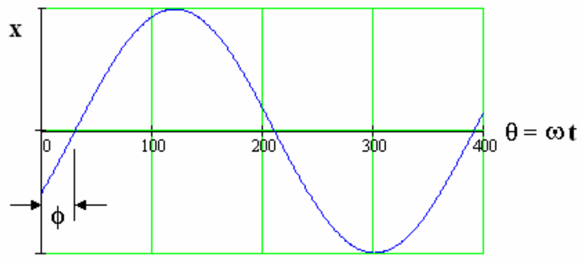
It follows that since $\theta = \omega t$ then $2\pi = \omega T$
Rearrange and $\omega = 2\pi/T$.
Substitute $T = 1/f$ and $\boldsymbol{\omega = 2\pi f}$



PHASE and DISPLACEMENT

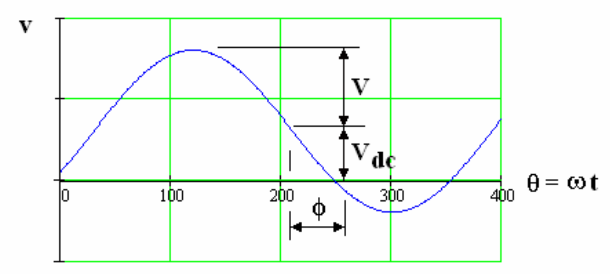
The sinusoidal graph is produced because we made $\theta = 0$ when $t = 0$. We could choose to make θ any value at $t = 0$. We would then write the equations as:

$x = A \sin(\theta + \phi)$ or $v = V \sin(\theta + \phi)$ where ϕ is the phase angle.
The plot shown has $x = 0$ at $\theta = 30^o$ so it follows that $(30 + \phi) = 0$ and so $\phi = -30^o$.



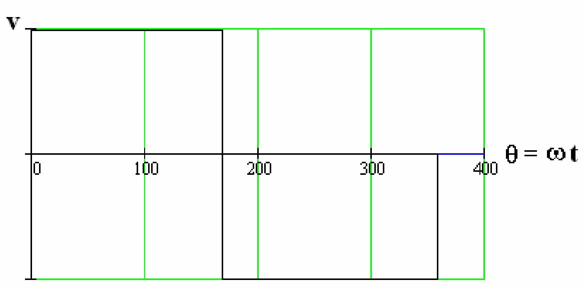
If we made $\phi = 90^o$ we would have a cosine plot.

Often periodic functions are not based about a mean of zero. For example an alternating voltage might be added to a constant (d.c.) voltage so that $V = V_{dc} + V \sin(\theta + \phi)$



SQUARE WAVES

Many periodic functions have other wave forms and one of the most common is the square wave produced by switching. This is common in digital electronics.



The mathematical equation for a square wave is not covered here.

SELF ASSESSMENT EXERCISE No.1

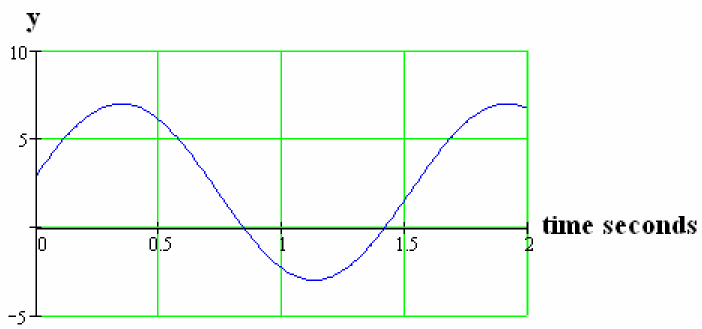
1. Mains electricity has a frequency of 50 Hz. What is the periodic time and angular frequency? (0.02 s and 314 rad/s)

2. An alternating current has a periodic time of 0.0025 s. What is the frequency? (400 Hz)

3. A alternating voltage has a peak to peak amplitude of 300 V and frequency of 50 Hz. What is the amplitude? (150 V)
   What is the voltage at t = 0.02 s? (16.4 V)

4. Determine the following from the graph shown.

   The amplitude.
   The offset displacement.
   The periodic time.
   The frequency.
   The angular frequency.
   The phase angle.



(Answers 5, 2, 1.57 s, 4 rad/s, 0.637 Hz, 0.2 radian or 11.5^o)

PHASOR DIAGRAMS

A phasor may be described as a vector rotating anticlockwise (positive direction). A mechanical phasor representing displacement has a length R and rotates at ω rad/s so this is in fact the Scotch Yoke described earlier. $x = R \sin(\omega t)$

An electrical phasor representing a sinusoidal voltage has a length V and rotates at ω rad/s.
$v = V_m \sin(\omega t)$ or $v = V_m \sin(2\pi f t)$

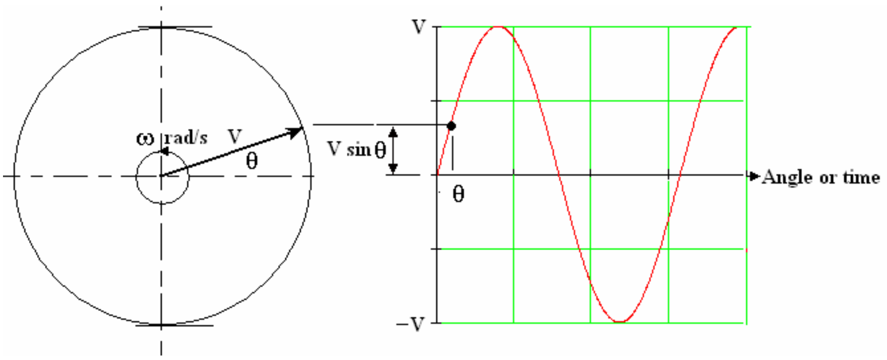
An electrical phasor representing current has a length I and rotates at ω rad/s.
$i = I_m \sin(\omega t)$ or $i = I_m \sin(2\pi f t)$

V_m and I_m are the amplitudes.

The diagram shows a phasor of length V rotating anticlockwise at ω rad/s. Starting from the horizontal position after a time t it will have rotated an angle $\theta = \omega t$.

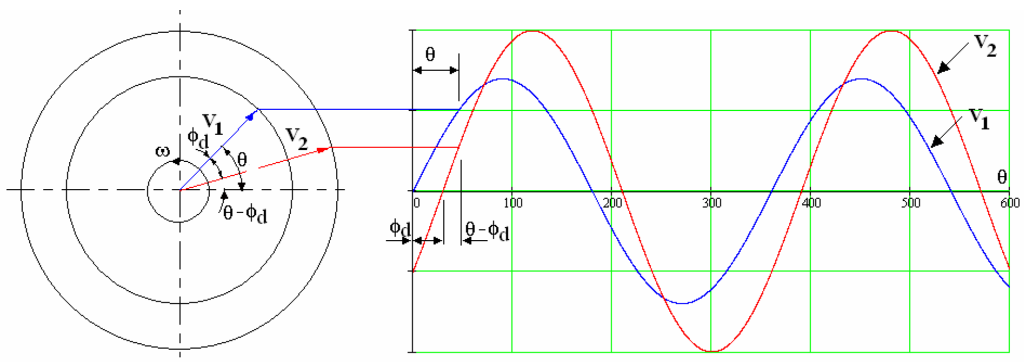
The vertical component of the phasor is $v = V \sin(\theta)$ (the m for max has been dropped)
This corresponds to the value of the sinusoidal graph at that angle.

When $\theta = \pi/2$ radian (90^o) the peak value is V so V is the amplitude or peak value (not the r.m.s.value).

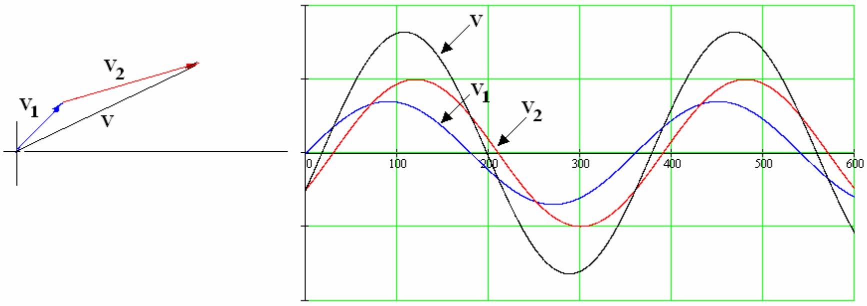


Phasors can be added or subtracted from each other in the same way as vectors. Consider two voltage phasors with the same angular frequency. The diagram shows two phasors V_1 and V_2 rotating at the same angular frequency but V_2 lags V_1 by angle ϕ_d. This is the PHASE DIFFERENCE BETWEEN THEM. We can express the vectors as:

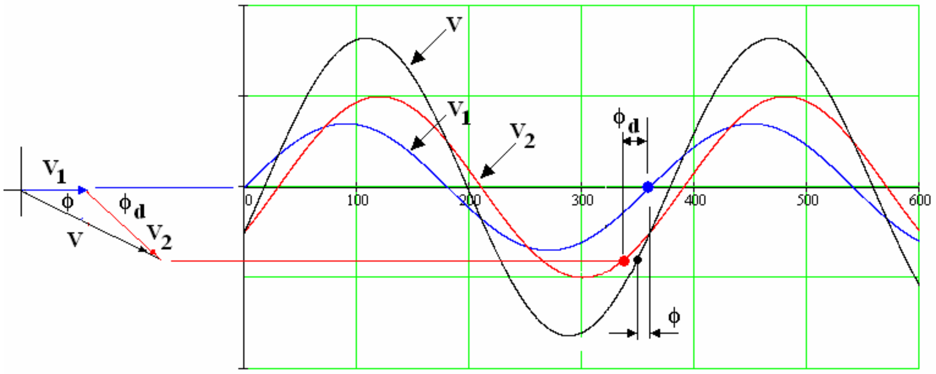
$V_1 = |V_1| \sin(\theta)$ and $V_2 = |V_2| \sin(\theta - \phi_d)$. The '| |' indicates the magnitude of the vector.



If we add the two vectors we do it as shown.



It is normal to draw one of the phasors horizontal as the phase difference is the same at all angles.



The easiest way to add vectors is to resolve them into horizontal and vertical components.

V_2 has a vertical component $|V_2| \sin \phi_d$ and a horizontal component $|V_2| \cos \phi_d$

The vertical component of V is $|V_2| \sin \phi_d$

The horizontal component of V is $|V_1|+ |V_2| \cos \phi_d$

WORKED EXAMPLE No. 1

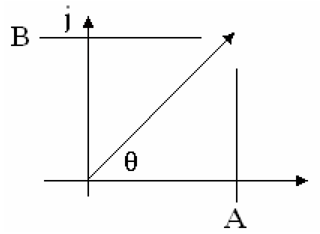
If $V_1 = 10 \sin (\theta)$ and $V_2 = 7 \sin (\theta - 30^o)$ what is the resultant voltage?

SOLUTION

Let $\theta = 0^o$ when we add them. (V_1 in the horizontal position).

The vertical component of V is $|V_2| \sin \phi_d = 7 \sin (-30^o) = -3.5$ (down).
The horizontal component of V is $|V_1| + |V_2| \cos \phi_d = 10 + 7 \sin 30^o = 16.06$
The resultant voltage is $\sqrt{}(-3.5^2+16.06^2) = 16.44$
The phase angle is $\phi = \tan^{-1}(-3.5/16.06) =-12.3^o$
We can express the phasor as $V = 16.44 \sin (\theta - 12.3^o)$

USE OF COMPLEX NUMBERS



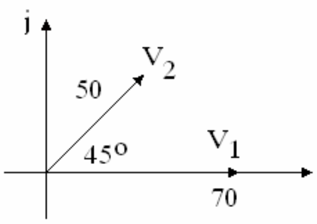
You might have realised that resolving a vector into vertical and horizontal components is the same as expressing it in terms of imaginary and real components on an Argand diagram. It follows that we can express a phasor as a complex number. The horizontal component is the real part and the vertical the imaginary part denoted with a j. Hence we can express a voltage as:
$V = A + jB \qquad \phi = \tan^{-1}(B/A)$ and $|V| = \sqrt{(A^2 + B^2)}$

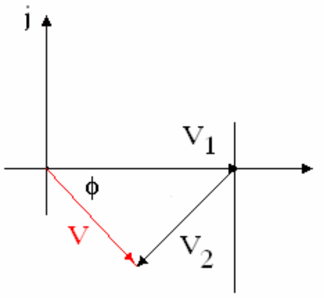
WORKED EXAMPLE No.2

Subtract the two alternating voltages $V_1 = 50 \sin(\theta)$ and $V_2 = 30 \sin(\theta + 45^o)$

SOLUTION

Put $\theta = 0 \quad V_1 = A_1 + j\,B_1 \qquad A_1 = 50 \qquad B_1 = 0$
$V_2 = A_2 + j\,B_2 \qquad A_2 = 30\cos(45) = 21.21 \qquad B_2 = 30\sin(45) = 21.21$
$V = V_1 - V_2 = A + jB = (50 - 21.21 + j(0 - 21.21)$
$V = 28.79 - j\,21.21$
$|V| = \sqrt{(A^2 + B^2)} = \sqrt{(28.79^2 + 21.21^2)} = 35.76$
$\phi = \tan^{-1}(B/A) = \tan^{-1}(-21.21/28.79) = -36.4^o$





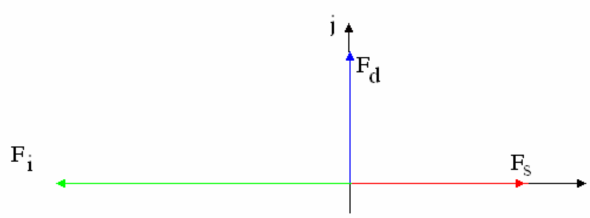
The resulting voltage is $V = 35.76 \sin(\theta - 36.4^o)$

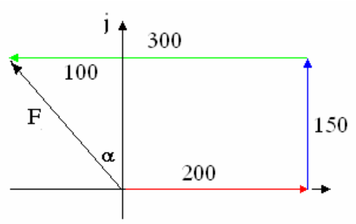
WORKED EXAMPLE No.3

When a mass, spring damper system is forced to oscillate the applied force is the sum of the spring force F_s, the damping force F_d and the inertia force F_i.
Given that $F_s = 200 \sin(\omega t)$ N, $F_d = 150 \sin(\omega t + 90^o)$ and $F_i = 300 \sin(\omega t + 180^o)$ express the resulting force as a phasor.

SOLUTION

Sketching the phasors shows this to be a simple problem.





The resulting force has a magnitude of $\sqrt{(100^2 + 150^2)} = 180.3$ N
$\alpha = \tan^{-1}(100/150) = 33.7^o$ hence $\phi = 123.7^o$
$F = 180.3 \sin(\omega t + 123.7^o)$

SELF ASSESSMENT EXERCISE No.2

1. Two sinusoidal voltages V_1 and V_2 are expressed by
   $V_1 = 240 \sin(\omega t)$ and $V_2 = 110 \sin(\omega t + 40^o)$
   What is the resulting voltage $V_1 + V_2$ expressed as a phasor?
   ($331.9 \sin(\omega t + 12.3^o)$)

2. Two sinusoidal voltages V_1 and V_2 are expressed by
   $V_1 = 240 \sin(\omega t)$ and $V_2 = 110 \sin(\omega t + 40^o)$
   What is the resulting voltage $V_1 - V_2$ expressed as a phasor?
   ($171 \sin(\omega t - 24.4^o)$)
   What is the resulting voltage expressed as a complex number?
   ($155.7 - j\,70.7$)

3. Two sinusoidal forces F_1 and F_2 are expressed by by $F_1 = 25 \sin(\omega t)$ and $F_2 = 12 \sin(\omega t + 30^o)$
   What is the resulting force $F_1 + F_2$ expressed as a phasor?
   ($35.9 \sin(\omega t + 9.6^o)$)
   What is the resulting force expressed as a complex number?
   ($33.4 + j6$)

4. When a mass, spring damper system is forced to oscillate the applied force is the sum of the spring force F_s, the damping force F_d and the inertia force F_i.

   Given that $F_s = 100 \sin(\omega t)$ N, $F_d = 30 \sin(\omega t + 90^o)$ and $F_i = 80 \sin(\omega t + 180^o)$ express the resulting force as a phasor.
   ($36 \sin\{\omega t + 56.3^o\}$)