Distance between points

The distance between P(x_1, y_1) and Q(x_2, y_2) is

$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$.

Symmetry Tests for Graphs :

1- symmetry about the x-axis:

If the point (x ,y) lies on the graph , then the point (x,-y) lies on the graph .

2- symmetry about the y-axis :

If the point (x ,y) lies on the graph , then the point (-x, y) lies on the graph .

3- symmetry about the origin :

If the point (x ,y) lies on the graph , then the point (-x,- y) lies on the graph .

Example :

The graph of $x^2 + y^2 = 1$ has all three of the symmetries

1- symmetry about the x-axis

(x ,y) on the graph

$\Rightarrow x^2 + y^2 = 1$

$\Rightarrow x^2 + (-y)^2 = x^2 + y^2 \quad (y^2 = (-y)^2)$

$\Rightarrow$(x,-y) lies on the graph .

2- symmetry about the y-axis

(x ,y) on the graph

$\Rightarrow x^2 + y^2 = 1$

$\Rightarrow (-x)^2 + y^2 = x^2 + y^2 \quad (x^2 = (-x)^2)$

$\Rightarrow$ (-x, y) lies on the graph .

3- symmetry about the origin :

(x ,y) on the graph

$\Rightarrow x^2 + y^2 = 1$

$\Rightarrow\ (-x)^2 + (-y)^2 = x^2 + y^2$

$\Rightarrow$ (-x,- y) lies on the graph .

Intercepts and More about graphing

We find the x-intercept by setting y equal to 0 in the equation and solving for x .

We find the y-intercept by setting x equal to 0 in the equation and solving for y .

Example :

Find the intercepts of the graph of the equation $y = x^2 - 1$.

Solution:

The x - intercepts $y = x^2 - 1$ (let y = 0)

$$\Rightarrow 0 = x^2 - 1 \Rightarrow x^2 = 1 \Rightarrow x = 1, -1$$

The y- intercepts $y = x^2 - 1$ (let x = 0)

$$\Rightarrow y = (0)^2 - 1 \Rightarrow y = -1$$

Exercises :

(1) Find the distance between the given points

(a) (1,0) and (0,1)

(b) (2,4) and (-1,0)

(2) Find the intercepts and test the symmetries their graph :

(a) $x = 1 - y^2$ (b) $4x^2 + 4y^2 = 1$.

Slope ,and Equations for lines :

Increments

Definition:

When a particle moves from (x_1, y_1) to (x_2, y_2) ,the increments are

$\Delta x = x_2 - x_1$ and $\Delta y = y_2 - y_1$.

Definition :

The slope of a nonvertical line is

$$m = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1} .$$

- The slope of a nonvertical line is the tangent of its angle of inclination .

$$m = \frac{\Delta y}{\Delta x} = \tan \Phi$$

Definition:

The equation $y - y_1 = m(x - x_1)$ is the point –slope equation of the line that passes through the point (x_1, y_1) with slope m .

Example :

Write an equation for the line that passes through the point (2 ,3) with slope $-3/2$.

Solution :

$$y - y_1 = m(x - x_1)$$
$$y - 3 = (-3/2)(x - 2)$$
$$y = \frac{-3}{2}x + 6$$

Example :

Write an equation for the line through (-2 ,-1) and (3 ,4) .

Sol:

We first calculate the slope

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{4 - (-1)}{3 - (-2)} = \frac{5}{5} = 1 \quad .$$

The (x_1, y_1) in the point – slope equation can be with

(x_1, y_1) = (-2 ,-1)	(x_1, y_1) = (3 ,4)
$y - (-1) = 1.(x - (-2))$	$y - 4 = 1.(x - 3)$
$y + 1 = x + 2$	$y - 4 = x - 3$
$y = x + 1$	$y = x + 1$

Definition :

The equation $y = mx + b$ is the slope – intercepts equation of the line with slope m and y- intercept b .

Remarks :

(1) vertical lines have no slope .
(2) Horizontal lines have slope 0 .
(3) For lines that are neither horizontal nor vertical it is handy to remember :

(a) they are parallel $\Leftrightarrow m_1 = m_2$.

(b) they are perpendicular $\Leftrightarrow m_2 = -1/m_1$.

The distance from a point to a line

To calculate the distance from a point P(x_1, y_1) to a line L ,we find the point Q(x_2, y_2) at the foot of the perpendicular from P to L and calculate the distance from P to Q .

Example :

Find the distance from the point P(2,1) to the line L : $y = x + 2$

Sol:

Step 1: We find an equation for the line L' through P(2 ,1) perpendicular to L . The slope of L : $y = x + 2$ is $m = 1$. The slope of L' is therefore $m' = -1/1 = -1$. We set (x_1, y_1) =(2,1) and $m = -1$ to find L' is $y - 1 = -1(x - 2)$

$$\Rightarrow \quad y = -x + 3 \quad .$$

Step 2 :

Find the point Q by solving the equation for L and L' simultaneously . To find the x-coordinate of Q , we equate the two expressions for y :

$$-x + 3 = x + 2$$

$$\Rightarrow \quad x = 1/2 \quad .$$

We obtain if $x = 1/2$ and by substituting in $y = x + 2$,we get $y = 5/2$. The coordinates of Q are (1/2 , 5/2) .

Step 3: We calculate the distance between P(2,1) and Q (1/2 , 5/2)

$$d = \sqrt{\left(2 - \frac{1}{2}\right)^2 + \left(1 - \frac{5}{2}\right)^2} = \frac{\sqrt{18}}{4} = \frac{3\sqrt{2}}{2}$$

Exercises :

(1) Find the slope if
(a) A(-2 ,-1) , B (1 ,-2) (b) A (1,2) , B (1, -3)

(2) Write an equation for the line that passes through the point P and has the slope m .
(a) P(-1 ,1) , m = 1 (b) P (1 ,-1) ,m = -1
(c) P (4 ,0) , m = -2

(3) Find an equation for the line through P perpendicular to L . Then the distance from P to L .
(a) P (0 , 0) , L : $y = -x + 2$.
(b) P (1 ,2) , L : $x + 2y = 3$

(c) P (-2 , 2) , L : $2x + y = 4$

Functions

Definition : (definition of function)

$\forall x \in D \quad \exists y \in R$ such that $f(x) = y$,where D is the domain set And R is the range set .

Example :

Find the domain and range set for the function

Function	Domain	Range
$y = x^2$	$-\infty < x < \infty$	$0 \le y$
$y = \sqrt{1 - x^2}$	$-1 \le x \le 1$	$0 \le y \le 1$
$y = \dfrac{1}{x}$	$x \ne 0$	$y \ne 0$
$y = \sqrt{x}$	$0 \le x$	$0 \le y$
$y = \sqrt{4 - x}$	$x \le 4$	$0 \le y$

Even and odd Functions

Definition :

A function $y = f(x)$ is an even function of x if $f(-x) = f(x)$ for every x in the function `s domain . It is an odd function of $x$ if $f(-x) = -f(x)$ for every $x$ in the function `s domain .

Example :

$f(x) = x^2$ Even function : $(-x)^2 = x^2$ for all x

$f(x) = x$ Odd function : $(-x) = -x$ for all x

Composition of functions

Def :

We say that the function $f(g(x))$ is the composite of g and f . Thus , the value of fog at x is $f(g(x)) = (fog)(x)$.

Example :

Find a formula for $f(g(x))$ if $g(x) = x^2$ and $f(x) = x - 7$. Then find the value of $f(g(2))$.

Sol :

$f(x) = x - 7$

$\Rightarrow f(g(x)) = g(x) - 7 = x^2 - 7$.

We then find the value of $f(g(2))$ by substituting 2 for x :

$f(g(2)) = 2^2 - 7 = -3$.

Equation for Circle in the Plane

Def :

A circle is the set of points in a plane whose distance from a given fixed point in the plane is a constant . The fixed point is the center of the circle . The constant distance is the radius of the circle .

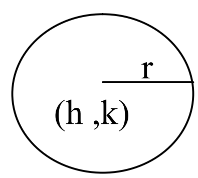
The standard equation for the circle of radius a centered at the point (h ,k) is

$$(x-h)^2 + (y-k)^2 = a^2$$

Example:

Find the center and radius of the circle

$$(x-1)^2 + (y+5)^2 = 3$$



Sol:

Comparing $(x-h)^2 + (y-k)^2 = a^2$

$\Rightarrow h = 1$, $k = -5$, $a^2 = 3 \Rightarrow a = \sqrt{3}$.

The standard equation for the circle of radius a centered at the origin

$$x^2 + y^2 = a^2$$

Equation for Parabolas

Definition :

A parabola is the set of points in a plane that are equidistance from a given fixed point and fixed line . The fixed point is the focus of the parabola . The fixed line is called the directrix of the parabola . The standard equation for the parabola is $y = \frac{x^2}{4p}$.

Example :

Find the focus and directrix of the parabola

Sol:

Find the value of p in the standard equation $y = \frac{x^2}{8}$ is $y = \frac{x^2}{4p}$

$\Rightarrow 4p = 8 \Rightarrow p = 2$.

Then find the focus and directrix for this value of p

Focus $(0\ ,\ p)$ or $(0,2)$

Directrix $(\ y = -p\ )$ or $y = -2$.

A Review of Trigonometric Function

Remark :

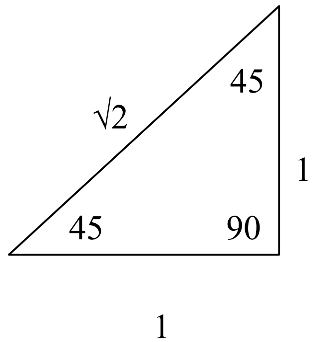
$180^{\circ} = \pi$ radians $\Rightarrow 1\deg ree = \frac{\pi}{180}$

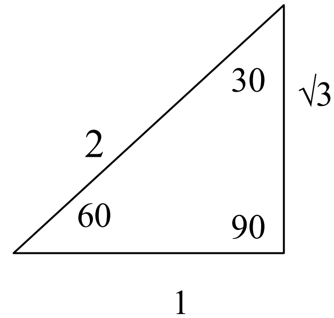
change 45° to radians $45.\frac{\pi}{180} = \frac{\pi}{4} rad$

change 90° to radians $90.\frac{\pi}{180} = \frac{\pi}{2} rad$

change $\frac{\pi}{6} radians$ to degree $\frac{\pi}{6}.\frac{180}{\pi} = 30^{\circ}$

change $\frac{\pi}{3} radians$ to degree $\frac{\pi}{3}.\frac{180}{\pi} = 60^{\circ}$





Example :

An acute angle whose vertex lies at the center of a circle of radius 6 subtends an arc of length 2π .Find the angle `s radian measure .

Sol: fig #

$$\theta = \frac{s}{r} = \frac{2\pi}{6} = \frac{\pi}{3} \ .$$

The angle measure is $\frac{\pi}{3}$ radians .

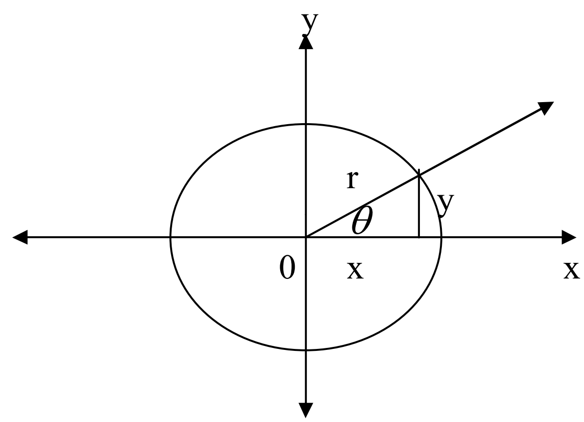


Fig *

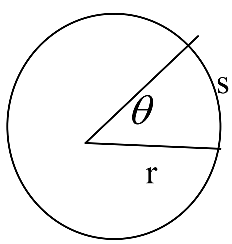


Fig #

When an angle of measure θ is placed in standard position at the center of a circle of radius r , the six basic trigonometric functions of θ are defined : figure *

$\sin\theta = \frac{y}{r}$, $\cos\theta = \frac{x}{r}$, $\tan\theta = \frac{y}{x}$,

$\csc\theta = \frac{r}{y}$, $\sec\theta = \frac{r}{x}$, $\cot\theta = \frac{x}{y}$

$\tan\theta = \dfrac{\sin\theta}{\cos\theta}$, $\csc\theta = \dfrac{1}{\sin\theta}$, $\sec\theta = \dfrac{1}{\cos\theta}$

$\cot\theta = \dfrac{1}{\tan\theta}$.

By (Pythagorean theorem)

$$x^2 + y^2 = r^2$$

$$\Rightarrow \cos^2\theta + \sin^2\theta = \frac{x^2}{r^2} + \frac{y^2}{r^2} = \frac{x^2 + y^2}{r^2} = 1$$

$\Rightarrow \cos^2\theta + \sin^2\theta = 1$ true for all θ

The coordinates of the point $P(x, y)$ can be expressed in terms of r and θ as

$x = r\cos\theta$, $y = r\sin\theta$

Note that if $\theta = 0$ then $x = r$ and $y = 0$, so

$\cos 0 = 1$, $\sin 0 = 0$

If $\theta = \dfrac{\pi}{2}$ then $x = 0$ and $y = r$, so

$\cos\dfrac{\pi}{2} = 0$, $\sin\dfrac{\pi}{2} = 1$.

* $\cos(-\theta) = \cos\theta$ the cosine is the even function .
* $\sin(-\theta) = -\sin\theta$ the sine is the odd function .

$\sec(-\theta) = \sec\theta$, $\tan(-\theta) = -\tan\theta$

$$\# \begin{cases} \cos(A \pm B) = \cos A\cos B \mp \sin A\sin B \\ \sin(A \pm B) = \sin A\cos B \pm \cos A\sin B \\ \cos 2\theta = \cos^2\theta - \sin^2\theta \\ \sin 2\theta = 2\sin\theta\cos\theta \\ \cos^2\theta = \dfrac{1 + \cos 2\theta}{2} \\ \sin^2\theta = \dfrac{1 - \cos 2\theta}{2} \end{cases}$$

The Absolute value function

The absolute value of a number x ,denoted by $|x|$ is defined by the formula

$$|x| = \begin{cases} x & \text{if } x \geq 0 . \\ -x & \text{if } x < 0 . \end{cases}$$

* Note that $|x| = \sqrt{x^2}$.

Example :
$$|2| = \sqrt{2^2} = 2$$
$$|-2| = \sqrt{(-2)^2} = 2$$

Properties of absolute value

1- $|-a| = |a|$.

2- $|ab| = |a|\ |b|$.

3- $\left|\frac{a}{b}\right| = \frac{|a|}{|b|}$

Example :

$$|-\sin x| = |\sin x| \quad , \quad |-2(x+5)| = |2||x+5| = 2|x+5|$$

$$\left|\frac{3}{x}\right| = \frac{|3|}{|x|} = \frac{3}{|x|} \quad .$$

4- $|a+b| \leq |a| + |b| \quad \forall a, b$

Ex :

$$|-3+5| = |2| = 2 < \ |-3| + |5| = 8$$
$$|3+5| = |8| = 8 = |3| + |5| \ .$$

* * If D is any positive number , then

$|a| < D \Leftrightarrow -D < a < D$.

$|a| \leq D \Leftrightarrow -D \leq a \leq D$.

Example :

$$|x-5| < 9$$

to $-9 < x - 5 < 9$

to $-9+5 < x < 9+5$

or $-4 < x < 14$.

Ex :

1. $\left|\frac{2x}{3}\right| \leq 1$ H.W

2. $\left|\frac{x}{2}-1\right| \leq 1$ H.W

Example :

$$\left|5-\frac{2}{x}\right| < 1$$

Sol :

$$\left|5-\frac{2}{x}\right| < 1$$

$$\Rightarrow \quad -1 < 5-\frac{2}{x} < 1$$

$$\Rightarrow \quad -6 < \frac{-2}{x} < -4$$

$$\Rightarrow \quad 4 < \frac{2}{x} < 6$$

$$\Rightarrow \quad 2 < \frac{1}{x} < 3$$

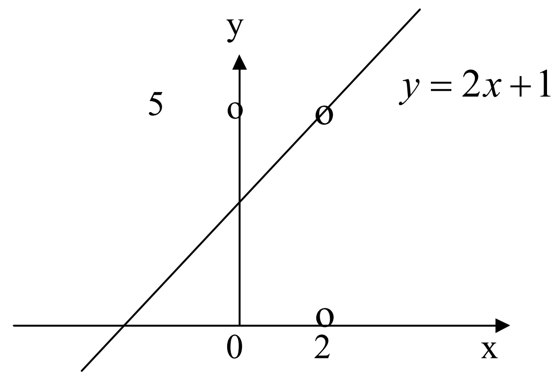
$$\Rightarrow \quad \frac{1}{3} < x < \frac{1}{2}$$.

Limits and Continuity

Def :

If the value of a function f of x approach the value L as x approaches c , we say f has limit L as x approaches c and we write $\lim_{x \to c} f(x) = L$.

Example : $\lim_{x \to 2}(2x+1) = 2*2+1 = 5$.



Ex : If f is the identity function $f(x) = x$, then for any value of c $\lim_{x \to c} f(x) = \lim_{x \to c}(x) = c$.

Ex : If f is the constant function $f(x) = k$, k is constant value $\lim_{x \to c} f(x) = \lim_{x \to c}(k) = k$.

Ex :

$f(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0$ is any polynomial function , then

$\lim_{x \to c} f(x) = f(c) = a_n c^n + a_{n-1} c^{n-1} + \ldots + a_1 c + a_0$.

Ex :

If $f(x)$ and $g(x)$ are polynomials , then $r(x)$ is called rational function $r(x) = \dfrac{f(x)}{g(x)}$

$\lim_{x \to c} r(x) = \lim_{x \to c} \dfrac{f(x)}{g(x)} = \dfrac{f(c)}{g(c)}$ (provided $g(c) \neq 0$).

Ex : Find

$$\lim_{x\to 3} x^2(2-x) = \lim_{x\to 3} x^2 . \lim_{x\to 3}(2-x) = -9 \quad .$$

Ex : Find

$$\lim_{x\to 2}\frac{x^2+2x+4}{x+2} = \frac{\lim_{x\to 2}(x^2+2x+4)}{\lim_{x\to 2}(x+2)} = \frac{12}{4} = 3 \quad .$$

Ex : Find

$$\lim_{x\to 2}\frac{x^3-8}{x^2-4} = L$$

$$\frac{x^3-8}{x^2-4} = \frac{(x-2)(x^2+2x+4)}{(x-2)(x+2)} = \frac{x^2+2x+4}{x+2}$$

$$L = \lim_{x\to 2}\frac{x^2+2x+4}{x+2} = \frac{12}{4} = 3 \quad .$$

Ex :

$$\lim_{x\to 5}\frac{x^2-25}{3(x-5)}$$ H . W

Relation ship between One –sided and two –sided limits

A function f of x has a limit as x approaches c iff the right –hand and left –hand limits at c exist and are equal .

$$\lim_{x\to c} f(x) = L \Leftrightarrow \lim_{x\to c^+} f(x) = L$$

$$\lim_{x\to c^-} f(x) = L$$

Example : Show that $\lim_{x\to 0}|x| = 0$.

Sol : We prove that $\lim_{x\to 0}|x| = 0$ by showing that the right –hand and left –hand limits are both 0 :

$$\lim_{x\to 0^+}|x| = \lim_{x\to 0^+} x = 0 \quad (\ |x| = x \ \text{ if } \ x > 0 \ )$$

$$\lim_{x\to 0^-}|x| = \lim_{x\to 0^-} -x = 0 \quad (\ |x| = -x \ \text{ if } \ x < 0 \ )$$

Theorem : (The sandwich theorem)

Suppose that $g(x) \le f(x) \le h(x)$ for all $x \ne c$ in some interval about c and that

$$\lim_{x\to c} g(x) = \lim_{x\to c} h(x) = L \quad .$$

Then $\lim_{x\to c} f(x) = L$.

Example : by using the sandwich theorem , then $\lim_{x\to 0}\dfrac{\sin x}{x} = 1$.

Limits Involving Infinity

Example :

The function $f(x) = \dfrac{1}{x}$ is defined for all real numbers except $x = 0$. We summarize these facts by saying :

(a) As x tends to ∞ , $\dfrac{1}{x}$ approaches to 0 .

(b) As x approaches to 0 from the right , $\dfrac{1}{x}$ tends to ∞ .

(c) As x approaches to 0 from the left , $\dfrac{1}{x}$ tends to $-\infty$.

(d) As x tends to $-\infty$, $\dfrac{1}{x}$ approaches to 0 .

Ex :

$$\lim_{x\to\infty}\frac{-x}{7x+4} = \lim_{x\to\infty}\frac{-1}{7+4/x} = \frac{-1}{7+0} = \frac{-1}{7}$$

Ex :

$$\lim_{x\to-\infty}\frac{2x^2-x+3}{3x^2+5} = \lim_{x\to-\infty}\frac{2-(1/x)+(3/x^2)}{3+(5/x^2)}=\frac{2-0+0}{3+0}=\frac{2}{3}$$

Ex :

$$\lim_{x\to\infty}\frac{5x+2}{2x^3-1} = \lim_{x\to\infty}\frac{(5/x^2)+(2/x^3)}{2-(1/x^3)} = \frac{0+0}{2-0}=0$$

Ex :

$$\lim_{x\to\infty}\frac{-4x^3+7x}{2x^2-3x-10} \qquad \text{H . W}$$

Continuous Functions

The continuity Test

The function $y=f(x)$ is continuous at $x=c$ iff all three of the following statements are true :

1- $f(c)$ exists (c lies in the domain of f)

2- $\lim_{x\to c} f(x)$ exists .

3- $\lim_{x\to c} f(x) = f(c)$.

Ex:

The sine and cosine are continuous at every value of x .
When $x = 0$

$\lim_{x\to 0}\sin x = 0 = \sin 0$ and $\lim_{x\to 0}\cos x = 1 = \cos 0$.

Ex :

The function $y=\frac{1}{x}$ is continuous at every value of x except $x = 0$.
The function is not defined at $x = 0$. f is not continuous .

Ex :

The function $h=\frac{x+3}{x^2-3x-10}$ is continuous at every value of x except $x = 5$ and $x = -2$.

Theorem :

If f is continuous at c and g is continuous at $f(c)$, then the composite gof is continuous at c .

Ex : Show that the function $y = \left|\frac{x\sin x}{x^2+2}\right|$ is continuous at every value of x .

Sol :

The function y is the composite of the continuous function

$f(x) = \frac{x\sin x}{x^2+2}$ continuous $\forall x$,

and $g(x) = |x|$ is continuous $\forall x$, therefore gof is continuous at x .

Exercises :

(1) The function $f(x) = \frac{x^2-1}{x-1}$ when $x \neq 1$ and by $f(1) = 2$

Is f cont. at $x = 1$.

(2) what value should be assigned to a make the function

$$f(x) = \begin{cases} x^2 - 1 & x < 3 \\ 2ax & x \geq 3 \end{cases}$$

Cont . at $x = 3$?.

Derivatives

Definition :

The derivative of a function f is the function f' whose value at x is defined by the equation

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \qquad \ldots\ldots\ldots\ldots\ldots \quad (*)$$

The fraction $(f(x+h) - f(x))/h$ is the difference quotient for f at x .

The slope and Tangent

When the number $f'(x)$ exists it is called the slope of the curve $y = f(x)$ at x . The line through the point $(x, f(x))$ with slope $f'(x)$ is the tangent to the curve at x .

Ex :
The derivative of the function $f(x) = mx + b$ is the slope of the line $y = mx + b$.

Sol : The calculation takes four steps :

Step 1: write out $f(x)$ and $f(x+h)$:

$$f(x) = mx + b$$
$$f(x+h) = m(x+h) + b$$
$$= mx + mh + b$$

Step 2 :

$$f(x+h) - f(x) = mh$$

Step 3 :

Divide by h

$$\frac{f(x+h) - f(x)}{h} = \frac{mh}{h} = m \ .$$

Step 4 : Take the limit as $h \to 0$

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} m = m \ .$$

Ex : Find dy/dx if $y = \frac{1}{x}$.

Sol : we take $f(x) = \frac{1}{x}$, $f(x+h) = \frac{1}{x+h}$, and from the quotient

$$\frac{f(x+h) - f(x)}{h} = \frac{\frac{1}{x+h} - \frac{1}{x}}{h}$$

$$= \frac{1}{h} . \frac{x-(x+h)}{x(x+h)}$$

$$= \frac{1}{h} . \frac{-h}{x(x+h)} = \frac{-1}{x(x+h)}$$

We then take the limit as $h \to 0$:

$$\frac{dy}{dx} = \lim_{h \to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h \to 0} \frac{-1}{x(x+h)} = \frac{-1}{x(x+0)} = \frac{-1}{x^2} \quad .$$

Ex : Find $f'(x)$ if $f(x) = x^2 - 2x$ H . W

Ex : show that the derivative of $y = \sqrt{x}$ is $\frac{dy}{dx} = \frac{1}{2\sqrt{x}}$. H .W

Ex : Find an equation for the tangent to the curve $y = \sqrt{x}$ at $x = 4$

Sol :

The slope at $x = 4$ is the value of the function `s derivative there . by above example gives $\frac{dy}{dx} = \frac{1}{2\sqrt{x}}$. so the slope is

$$\left. \frac{1}{2\sqrt{x}} \right|_{x=4} = \frac{1}{2\sqrt{4}} = \frac{1}{4} \quad .$$

The tangent is the line through the point $P(4, \sqrt{4}) = (4, 2)$ with slope 1/4 :

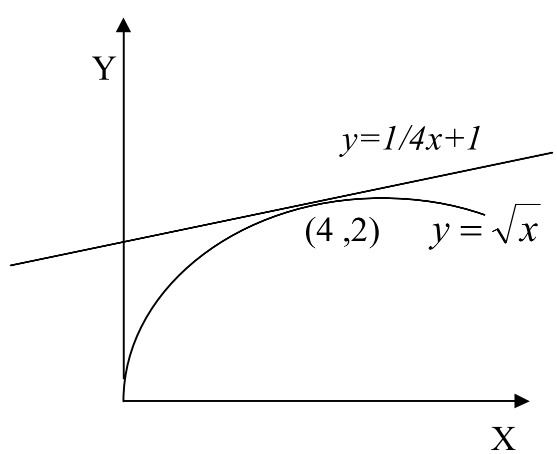
Point (4 , 2)

Slope 1/4

Equation $y - 2 = (1/4)(x-4)$

$$y = \frac{1}{4}x - 1 + 2$$

$$\therefore \quad y = \frac{1}{4}x + 1$$



Theorem : If f has a derivative at $x = c$, then f is continuous at $x = c$.

Remark :

A function has a (two sided)derivative at a point iff the function's right –hand and left –hand derivatives are defined and are equal

$\Rightarrow f(x) = y$ is differentiable on a closed interval [a , b] .

$$\lim_{h\to 0^+} \frac{f(a+h)-f(a)}{h} \quad \text{(right –hand derivative at } a \text{)}$$

$$\lim_{h\to 0^-} \frac{f(b+h)-f(b)}{h} \quad \text{(left –hand derivative at } b \text{) .}$$

Ex :

The function $y = |x|$ has no derivative at $x = 0$

Sol:

$$\lim_{h\to 0^+} \frac{|0+h|-|0|}{h} = \lim_{h\to 0^+} \frac{|h|}{h} = \lim_{h\to 0^+} 1 = 1 \quad \text{(because } |h| = +h \text{ , } h > 0 \text{)}$$

$$\lim_{h\to 0^-} \frac{|0+h|-|0|}{h} = \lim_{h\to 0^-} \frac{|h|}{h} = \lim_{h\to 0^-} \frac{-h}{h} = \lim_{h\to 0^-} -1 = -1 \quad \text{(because}$$

$|h| = -h$, $h < 0$) .

left –hand derivative $\neq$ right –hand derivative .

Differentiation Rules

1- If c is a constant , then

$$\frac{d}{dx} c = 0 \ .$$

2- If u is a diff. function of x and c is a constant ,then

$$\frac{d}{dx}(c\,x^n) = cn\,x^{n-1} \ .$$

3- If u is a diff. function of x and if v is a diff. function of x ,then

$$\frac{d}{dx}(u \pm v) = \frac{du}{dx} \pm \frac{dv}{dx} \quad .$$

Ex : Find $y^{(4)}$

$y = x^3 + 6$

Ex : H .W

1- Find $y^{(4)}$ $\quad y=x^4 + 12x$

2- Find y''' $\quad y=x^3 - 3x^2 + 2$

3- Find $y^{(4)}$ $\quad y = x^5 + 3x^3 - 2x^2 + 6$

4- - If u is a diff. function of x and if v is a diff. function of x ,then

$$\frac{d}{dx}(u.v) = u\frac{dv}{dx} + v\frac{du}{dx} \quad .$$

5- If u is a diff. function of x and if v is a diff. function of x and $v \neq 0$,then

$$\frac{d}{dx}(\frac{u}{v}) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2} \quad .$$

Ex : Find the derivative of $y = \frac{x^2 - 1}{x^2 + 1}$.

Derivative of Trigonometric functions

1- $\frac{d}{dx}\sin x = \cos x$.

2- $\frac{d}{dx}\cos x = -\sin x$

3- $\frac{d}{dx}\tan x = \sec^2 x$

4- $\frac{d}{dx}\cot x = -\csc^2 x$

5- $\frac{d}{dx}\sec x = \sec x \tan x$

6- $\frac{d}{dx}\csc x = -\csc x \cot x$

Ex : Find y' if

1- $y = x^2 - \sin x$

2- $y = \sin x . \sec x$ H .W

3- $y = 3x^2 + x \tan x$

Remark :

$(fog)' = f'_{(g(x))} . g'(x)$ by using the Chain Rule

And if y is a diff. function of u and u is a diff. function of x , then :

$$\frac{dy}{dx} = \frac{dy}{du} . \frac{du}{dx} \quad .$$

Ex : If $f(u) = \sin u$ and $u = g(x) = x^2 - 4$, find $(fog)'$ at $x = 2$.

Sol :

$$(fog)'_{x=2} = f'_{u=g(2)} . g'_{x=2} \quad = \frac{d}{dx}(\sin u)_{u=0} . \frac{d}{dx}(x^2 - 4)_{x=2}$$

$$= \cos u\big|_{u=0} . 2x\big|_{x=2} = 1.4 = 4$$

Ex :

Find dy/dx at $x = 0$ if $y = \cos u$ and $u = \frac{\pi}{2} - 3x$

H.W .

Ex :

$$\frac{d}{dx}\sin^5 x = 5\sin^4 x \frac{d}{dx}(\sin x) = 5\sin^4 x \cos x \quad .$$

Ex:

1- $\frac{d}{dx}(2x+1)^{-3}$

2- $\frac{d}{dx}\tan\sqrt{x}$

3- $\frac{d}{dx}\cos^2 3x$

4- $\frac{d}{dx}\sin(-x)$

5- $\frac{d}{dx}\tan(\frac{1}{x})$

(1–5: H .W)

6- Express dy/dx in terms of x if
$y=u^3$ and $u=x^2-1$. H . W

Implicit Differentiation

Implicit Diff. Takes Four Steps

1- Differentiation both sides of the equation with respect to x .

2- Collect the terms with $\frac{dy}{dx}$ on one side of the equation .

3- Factor out $\frac{dy}{dx}$.

4- Solve for $\frac{dy}{dx}$ by dividing .

Ex : Find $\frac{dy}{dx}$ if $2y = x^2 + \sin y$

Sol : $2y = x^2 + \sin y$

$$\frac{d}{dx}(2y) = \frac{d}{dx}(x^2) + \frac{d}{dx}(\sin y) \qquad \text{[step 1]}$$

$$2\frac{dy}{dx} = 2x + \cos y \frac{dy}{dx}$$

$$2\frac{dy}{dx} - \cos y \frac{dy}{dx} = 2x \qquad \text{[step 2]}$$

$$(2 - \cos y)\frac{dy}{dx} = 2x \qquad \text{[step 3]}$$

$$\frac{dy}{dx} = \frac{2x}{2 - \cos y} \qquad \text{[step 4]}$$

Ex : Find the tangent and normal to the curve $x^2 - xy + y^2 = 7$ at the point (-1,2) .

Sol :

We first use implicit diff. to find dy/dx .

$$x^2 - xy + y^2 = 7$$

$$\frac{d}{dx}(7) = \frac{d}{dx}(x^2) - \frac{d}{dx}(xy) + \frac{d}{dx}(y^2)$$

$$2x-(x\frac{dy}{dx}+y\frac{dx}{dx})+2y\frac{dy}{dx}=0$$

$$(2y-x)\frac{dy}{dx}=y-2x$$

$$\frac{dy}{dx}=\frac{y-2x}{2y-x} \quad .$$

We then evaluate the derivative at $x=-1$, $y=2$,to obtain

$$\left.\frac{dy}{dx}\right|_{(-1,2)} = \left.\frac{y-2x}{2y-x}\right|_{(-1,2)} = \frac{4}{5} \ .$$

The tangent to the curve at $(-1,2)$ is

$$y-2=\frac{4}{5}.(x-(-1))$$

$$y=\frac{4}{5}x+\frac{14}{5} \quad .$$

The normal to the curve at $(-1,2)$ is

$$y-2=-\frac{5}{4}.(x-(-1))$$

$$y=-\frac{5}{4}x+\frac{3}{4} \quad .$$

Ex: Find d^2y/dx^2 if $2x^3-3y^2=7$ H .W

Ex :.

$$\frac{d}{dx}(x^{\frac{1}{2}})=\frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}}$$

↑ ↑

Function defined for $x \geq 0$ — derivative defined only $x > 0$

***Rolle*`s Theorem**

Suppose that $y = f(x)$ is cont. at every point of the closed interval $[\, a , b \,]$ and differentiable at every point of its interior $(a \ , b)$.
If $f(a) = f(b) = 0$, then there is at least one number c between a and b at which $f'(c) = 0$.

Ex :

The polynomial function $f(x) = \frac{x^3}{3} - 3x$ is cont. at every point of the interval $-3 \le x \le 3$ and differentiable at every point of the interval $-3 < x < 3$. since $f(-3) = f(3) = 0$, Rolle`s theorem says that f' must be zero at least once in the open interval between $a = -3$ and $b = 3$. In fact, $f'(x) = x^2 - 3$ is zero twice in this interval , once at $x = -\sqrt{3}$ and again at $x = \sqrt{3}$.

The Mean Value Theorem

If $y = f(x)$ is cont. at every point of the closed interval $[\, a , b \,]$ and differentiable at every point of its interior $(a \ , b)$, then there is at least one number c between a and b at which

$$f'(c) = \frac{f(b) - f(a)}{b - a} .$$

Ex :

The function $f(x) = x^2$ is cont. for $0 \le x \le 2$ and diff. for $0 < x < 2$. Since $f(0) = 0$ and $f(2) = 4$, the Mean Value Theorem says that at some point c in the interval the derivative $f'(x) = 2x$ must have the value $(4 - 0) / (2 - 0) = 2$.
Thus $2c = 2$ to get $c = 1$.

Def :

A function $f(x)$ defined throughout an interval I is said to increase on I , if , for any two points x_1 and x_2 in I, $x_2 > x_1 \Rightarrow f(x_2) > f(x_1)$

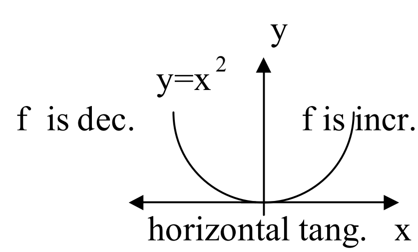
Similarly, f is said to decrease on I , if , for any two points x_1 and x_2 in I, $x_2 > x_1 \Rightarrow f(x_2) < f(x_1)$.

The Test for Increasing and Decreasing

f is increasing when $f'(x) > 0$. Also f is decreasing when $f'(x) < 0$,where $x \in I$.

Ex :.

The function $y = x^2$ decreases on $(-\infty, 0)$, where the derivative $y' = 2x$ is negative , and increase on $(0, \infty)$, where the derivative $y' = 2x$ is positive , $y' = 0$ and tangent to the curve is horizontal .



The second Derivative Test for concavity

The graph of $y = f(x)$ is concave down on any interval where $y'' < 0$.

The graph of $y = f(x)$ is concave up on any interval where $y'' > 0$.

Ex:.

The function $y = x^2$

The function $y = x^3$

Definition :

A point on the graph of a differentiable function where the concavity changes is called a point of inflection .

* At a point of inflection on the graph of a twice differentiable function $y'' = 0$.

Ex:.

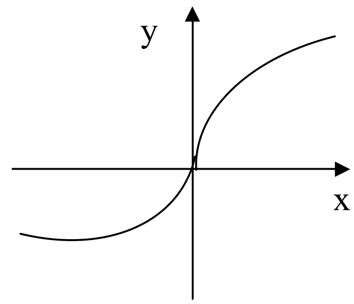
$y = \sin x$ at $x = \pi$ or $x = 0$.

Ex:.

No , $y = x^4$, $x = 0$.

Ex:.

$y = x^{\frac{1}{3}}$, $x = 0$



Steps in Graphing $y = f(x)$

1- Find y' and y''

2- Find where y' is positive , negative , and zero .

3- Find where y'' is positive , negative , and zero .

4- Make a summary table .

5- Draw the graph .

Ex:.

Graph the function $y = x^3 - 3x^2 + 4$

* x = -1 ,3

The second Derivative Test for local maxima and local minima

If $f'(c)=0$ and $f''(c) < 0$, then f has a local maximum at $x=c$.

If $f'(c)=0$ and $f''(c) > 0$, then f has a local minimum at $x=c$.

Def:

The average velocity of a body moving along a line from position $s = f(t)$ to position $s = f(t+\Delta t)$ is

$$v = \text{displacement / travel time} = \frac{\Delta s}{\Delta t} = \frac{f(t+\Delta t) - f(t)}{\Delta t} \quad .$$

The velocity is the derivative of position . If the position function of a body moving along a line is $s = f(t)$, the body `s velocity at time t is $v(t) = \frac{ds}{dt} = \lim_{\Delta t \to 0} \frac{f(t+\Delta t) - f(t)}{\Delta t}$ **.**

Ex :

If $s = f(t) = 160t - 16t^2$. Find the velocity at 256 ft .

Sol:

$$v(t) = \frac{ds}{dt} = \frac{d}{dt}(160t - 16t^2) = 160 - 32t \quad \text{ft/ sec .}$$

$s(t) = 160t - 16t^2 = 256$

$16t^2 - 160t + 256 = 0$

$16(t-2)(t-8) = 0$

$t = 2\sec$, $t = 8\sec$.

$v(2) = 160 - 32(2) = 96$ **ft/ sec**

$v(8) = -96$ **ft/ sec** .

Def:

Speed is the absolute value of velocity .

For example above speed is $|-96| = 96$ **ft/ sec** .

Def:

Acceleration is the derivative of velocity .

If a body `s position at time $t$ is $s = f(t)$ , then the body `s acceleration at time t is

$$a = \frac{dv}{dt} = \frac{d^2 s}{dt^2} \quad .$$

For example above is

$$a = \frac{dv}{dt} = \frac{d}{dt}(160 - 32t) = -32 \quad \text{ft/ sec}^2 \text{ .}$$

Integration

Def:

The definite integral of a function $f(x)$ over an interval $[a,b]$ Is usually denoted by $\int_a^b f(x)dx$, the number a and b are limits of integration a lower limit , b upper limit of integration .

Def:

If $y = f(x)$ is nonnegative and integrable over a closed interval $[a,b]$ then the integral of f from a to b is the area of the region between the graph of f and the $x-axis$ from a to b .

Ex:

Find the value of the integral $\int_{-2}^{2} \sqrt{4-x^2}dx$.

Sol:

See that the graph is a semicircle of radius 2 .The area between the semicircle and the $x-axis$ is

Area = $\frac{1}{2}.\pi r^2 = \frac{1}{2}\pi(2)^2 = 2\pi$.

Area is also the value of the integral of f from -2 to 2 ,

$$\int_{-2}^{2} \sqrt{4-x^2}dx = 2\pi .$$

Area = $-\int_a^b f(x)dx$ **when** $f(x) \le 0$.

Ex:

$y = x^2 - 1$ $\qquad\qquad$ $y = \cos x$

$x = -1,1$ $\qquad\qquad$ $x = \frac{\pi}{2}, \frac{3\pi}{2}$

*** If an integrable function $y = f(x)$ has both positive and negative values on an interval $[a,b]$.**

$$\int_a^b f(x)dx = \text{(area above } x-axis \text{) - (area below } x-axis \text{)}$$

Ex:

$$\int_0^{2\pi} \sin x dx = 0$$

Theorem : (The integral Evaluation theorem) (second Fundamental theorem of calculus)

If f is cont. at every point of $[a,b]$ and F is any anti derivative of f on $[a,b]$, then $\int_a^b f(x)dx = F(b) - F(a)$.

Ex:

$$\int_0^2 x^2 dx$$

Ex:

$$\int_{-2}^2 (4 - x^2)dx$$

Steps for Finding Area when the function has positive and negative values on $[a,b]$

Step 1 :

Find the points where $f = 0$

Step 2 :

Use the zeros of f to partition $[a,b]$ into subintervals .

Step 3 :

Integrate f over each subinterval .

Step 4 :

Add the absolute values of the results .

Ex:

Find the area of the region between the $x-axis$ **and the curve** $y=x^3-4x$, $-2\le x\le 2$.

Sol:

Step 1: The zeros of y **. To find where** y **is zero .**

$$y=x^3-4x=x(x^2-4)=x(x-2)(x+2)$$

The zeros occur at $x=-2{,}0$ **and** 2

Step 2 :

The intervals of integration . The point $x=-2{,}0$ **and** 2 **partition** $[-2,2]$ **into two subinterval** $[-2,0]$ **,** $[0,2]$ **.**

Step 3 :

$$\int_{-2}^{0}(x^3-4x)dx=\left[\frac{x^4}{4}-2x^2\right|_{-2}^{0}=4$$

$$\int_{0}^{2}(x^3-4x)dx=\left[\frac{x^4}{4}-2x^2\right|_{0}^{2}=-4$$

Step 4 :

Area of region $=|4|+|-4|=4+4=8$.

Indefinite Integrals

To find $\int f(x)dx$

Step 1 :

Find an antiderivatives $F(x)$ **of** $f(x)$

Step 1 :

Add c

$$\int f(x)dx=F(x)+c$$

Ex:

$$\int(x^2+5)dx=\frac{x^3}{3}+5x+c \ .$$

1- $\int \sin mx dx = -\frac{\cos mx}{m} + c$ 2- $\int \cos mx dx = \frac{\sin mx}{m} + c$

3- $\int \sec^2 x dx = \tan x + c$ 4- $\int \csc^2 x dx = -\cot x + c$

5- $\int \sec x \tan x dx = \sec x + c$ 6- $\int \csc x \cot x dx = -\csc x + c$

Ex: 1- $\int x^5 dx$ 2- $\int \cos \frac{x}{2} dx$

Ex: $\int \sin^2 x dx$ **Ex**: $\int \cos^2 x dx$

Ex: $\int (x+2)^5 dx$

Ex:

Evaluate $\int \frac{dx}{\sqrt{9-x^2}}$

Let $\sqrt{9-x^2} = \sqrt{9(1-\frac{x^2}{9})} = 3\sqrt{1-(\frac{x}{3})^2}$

,then $\int \frac{dx}{\sqrt{9-x^2}} = \int \frac{dx}{3\sqrt{1-(\frac{x}{3})^2}}$

We now substitute $u = \frac{x}{3} \qquad du = \frac{dx}{3} \Rightarrow dx = 3du$

$$\int \frac{dx}{\sqrt{9-x^2}} = \int \frac{dx}{3\sqrt{1-(\frac{x}{3})^2}} = \int \frac{3du}{3\sqrt{1-u^2}} = \sin^{-1} u + c = \sin^{-1}(\frac{x}{3}) + c$$

Ex:

Evaluate $\int \frac{x^2 dx}{\sqrt{1-x^6}}$ H.W

Hyperbolic Functions

1- $\sinh x = \frac{e^x - e^{-x}}{2}$

2- $\cosh x = \frac{e^x + e^{-x}}{2}$

3- $\tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}}$

4- $\coth x = \frac{e^x + e^{-x}}{e^x - e^{-x}}$

5- $\sec hx = \frac{1}{\cosh x} = \frac{2}{e^x + e^{-x}}$

6- $\csc hx = \frac{1}{\sinh x} = \frac{2}{e^x - e^{-x}}$

7- $\sinh 2x = 2\sinh x \cosh x$

8- $\cosh 2x = \cosh^2 x + \sinh^2 x$

9- $\cosh^2 x = \dfrac{\cosh 2x + 1}{2}$, $\sinh^2 x = \dfrac{\cosh 2x - 1}{2}$

10- $\cosh^2 x - \sinh^2 x = 1$

11- $\tanh^2 x = 1 - \sec h^2 x$

12- $\coth^2 x = 1 + \csc h^2 x$

- ***Derivative and Integrals of Hyperbolic Functions***

1- $\dfrac{d}{dx}(\sinh u) = \cosh u.\dfrac{du}{dx}$

2- $\dfrac{d}{dx}(\cosh u) = \sinh u.\dfrac{du}{dx}$

3- $\dfrac{d}{dx}(\tanh u) = \sec h^2 u.\dfrac{du}{dx}$

4- $\dfrac{d}{dx}(\coth u) = -\csc h^2 u.\dfrac{du}{dx}$

5 - $\dfrac{d}{dx}(\sec hu) = -\tanh u \sec hu.\dfrac{du}{dx}$

6- $\dfrac{d}{dx}(\csc hu) = -\coth u \csc hu.\dfrac{du}{dx}$

7- $\int \sinh u du = \cosh u + c$

8- $\int \cosh u du = \sinh u + c$

9- $\int \sec h^2 u du = \tanh u + c$

10- $\int \csc h^2 u du = -\coth u + c$

11- $\int \sec hu \tanh u du = -\sec hu + c$

12- $\int \csc hu \coth u du = -\csc hu + c$

Ex:

$$\int \coth 5x dx = \int \frac{\cosh 5x}{\sinh 5x} dx = \frac{1}{5}\int \frac{du}{u} \qquad (u = \sinh 5x)$$

$$= \frac{1}{5}\ln|u| + c = \frac{1}{5}\ln|\sinh 5x| + c$$

Ex: $\int_0^1 \sinh^2 x dx$

Ex: $\int_0^{\ln 2} 4e^x \sinh x dx$ **H .W**

Inverse of the hyperbolic functions

$1 - \operatorname{sech}^{-1} x = \cosh^{-1}\frac{1}{x}$, $\operatorname{csch}^{-1} x = \sinh^{-1}\frac{1}{x}$,

$\coth^{-1} x = \tanh^{-1}\frac{1}{x}$.

Derivative :

1- $\frac{d(\sinh^{-1} u)}{dx} = \frac{1}{\sqrt{1+u^2}}.\frac{du}{dx}$

2- $\frac{d(\cosh^{-1} u)}{dx} = \frac{1}{\sqrt{u^2-1}}.\frac{du}{dx}$, $u > 1$

3- $\frac{d(\tanh^{-1} u)}{dx} = \frac{1}{1-u^2}.\frac{du}{dx}$, $|u| < 1$

4- $\frac{d(\coth^{-1} u)}{dx} = \frac{1}{1-u^2}.\frac{du}{dx}$, $|u| > 1$

5- $\frac{d(\operatorname{sech}^{-1} u)}{dx} = \frac{-1}{u\sqrt{1-u^2}}.\frac{du}{dx}$, $0 < u < 1$

6- $\frac{d(\sinh^{-1} u)}{dx} = \frac{-1}{|u|\sqrt{1+u^2}}.\frac{du}{dx}$, $u \neq 0$

Ex: show that **if** u **is a diff. func. of** x **,** $u > 1$ **,**

$$\frac{d(\cosh^{-1} u)}{dx} = \frac{1}{\sqrt{u^2 - 1}} \cdot \frac{du}{dx}$$

Sol:

$$y = \cosh^{-1} x$$

$$x = \cosh y$$

$$1 = \sinh y . \frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{1}{\sinh y} = \frac{1}{\sqrt{\cosh^2 y - 1}} = \frac{1}{\sqrt{x^2 - 1}}$$

(since $x > 1$ **and** $y > 0$ **and** $\sinh y > 0$ **)**

By using chain rule ,

$$\frac{d(\cosh^{-1} u)}{dx} = \frac{1}{\sqrt{u^2 - 1}} \cdot \frac{du}{dx}$$

Integrals:

1- $\int \frac{du}{\sqrt{1 + u^2}} = \sinh^{-1} u + c$

2- $\int \frac{du}{\sqrt{u^2 - 1}} = \cosh^{-1} u + c \quad , \quad u > 1$

3- $\int \frac{du}{1 - u^2} = \begin{cases} \tanh^{-1} u + c & \text{if } |u| < 1 \\ \coth^{-1} u + c & \text{if } |u| > 1 \end{cases}$

4- $\int \frac{du}{u\sqrt{1 - u^2}} = -\sec h^{-1}|u| + c = -\cosh^{-1} \frac{1}{|u|} + c$

5- $\int \frac{du}{u\sqrt{1 + u^2}} = -\csc h^{-1}|u| + c = -\sinh^{-1} \frac{1}{|u|} + c$

Ex: Evaluate $\int_0^2 \frac{2dx}{\sqrt{1+4x^2}}$

Ex: Evaluate

$$\int \frac{(x-2)^3}{x^2-4}dx$$

Sol:

$$\frac{(x-2)^3}{x^2-4} = \frac{(x-2)^3}{(x-2)(x+2)} = \frac{x^2-4x+4}{x+2}$$

$$\frac{x^2-4x+4}{x+2} = x-6+\frac{16}{x+2}$$

$x-6$

$x+2 \,\overline{)\, x^2-4x+4}$

x^2+2

$-6x+4$

$-6x-12$

$+16$

Therefore ,

$$\int \frac{(x-2)^3}{x^2-4}dx \;= \int (x-6+\frac{16}{x+2})dx$$

$$= \frac{x^2}{2} - 6x + 16\ln|x+2| + c$$

Integration by parts:

The integration by part formula

$$\int udv = uv - \int vdu$$

Ex: Evaluate $\int x\cos xdx$

Sol:

Let $u = x$, $dv = \cos xdx$,

$du = dx$, $v = \sin x$

Then

$$\int x\cos xdx = x\sin x - \int \sin xdx$$

$$= x\sin x + \cos x + c \quad .$$

Ex: Evaluate $\int_0^1 xe^x dx$ H . W

Ex: **Evaluate** $\int \ln x dx$ **H .W**

Ex: **Evaluate** $\int x^2 e^{2x} dx$

Ex: **Evaluate** $\int e^x \cos x dx$

Sol:

Let $u = e^x \quad , \quad dv = \cos x dx$

$du = e^x dx \qquad v = \sin x$

$\int e^x \cos x dx = e^x \sin x - \int e^x \sin x dx$

Let $u = e^x \quad , \quad dv = \sin x dx$

$du = e^x dx \qquad v = -\cos x$

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$\Rightarrow 2\int e^x \cos x dx = e^x \sin x + e^x \cos x$

$$\int e^x \cos x dx = \frac{e^x \sin x + e^x \cos x}{2} + c$$

Ex: **Evaluate** $\int e^x x^2 dx$ **. By tabular integration**

Sol: with $f(x) = x^2$ **and** $g(x) = e^x$ **,we list :**

$f(x)$ and its derivative		$g(x)$ and its integrals
x^2	(+)	e^x
$2x$	(-)	e^x
2	(+)	e^x
0		e^x

$\int e^x x^2 dx = x^2 e^x - 2xe^x + 2e^x + c$

Ex: **Evaluate** $\int x^3 \sin x dx$ **. By tabular integration** **H .W**