**Accurate Frequency Estimation Method Based on Fast Subspace Decomposition Using Lanczos Algorithm**

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**Abstract:**

In this paper, the problem of estimating the frequency of noisy sinusoidal signal is addressed. Pisarenko harmonic decomposition method is discussed as a special case of Auto Regressive Moving Average (ARMA). A method for frequency estimation is presented without the necessary complete Eigen decomposition. The proposed method does not require the eigenvalue decomposition (EVD) or singular value decomposition (SVD) of the covariance matrix of the received signal; it is based on the Lanczos algorithm for improving the resolution of the frequency estimation. Therefore, computer simulations are included to demonstrate the effectiveness of the proposed method. The results show that there is a significant improvement has been achieved.

**Keyword: Pisarenko, subspace, Lanczos, frequency estimation, eigenvalue**

**الخلاصــــة:**

في هذا البحث، تم التصدي لمشكلة تقدير التردد لاشارة جيبة داخل الضوضاء. وتم شرح طريقة Pisarenko الأسلوب التوافقي التحللي باعتبارها حالة خاصة (ARMA). وتم تقديم طريقة لتخمين التردد دون التحلل الكامل للـ Eigen. الطريقة المقترحة لا يتطلب التحلل معامل التحول الخطى (EVD) أو التحلل المفرد القيمة (SVD) من مصفوفة التغاير للإشارة الواردة ، لأنه يقوم على الخوارزمية Lanczos لتحسين دقة تخمين التردد. لذلك، تم تضمين محاكاة حاسوبية للتدليل على فعالية الطريقة المقترحة. تظهرالنتائج ان هنالك تحسن كبير قد تحقق.

**1. Introduction:**

The Auto Regressive Moving Average ARMA has the excellent resolution properties, it is the most appropriate estimate for signals in white noise as given by [Hayes and Monson (1996)], that is because the effect of noise on the Auto Regressive frequency estimate is to produce a smoothed spectrum.

Pisarenko assumed that the ARMA model have a special symmetry where the Auto Regressive parameters are identical to the moving average parameter so, he joins the two best facilities in his model. Frequency estimation as given by [Kay (1988) and Stoica (1993)] has been universally addressed in signal processing literature. Conventional subspace method of frequency estimation using eigenvalue decomposition of covariance matrix of the received data matrix using these techniques, the computational complexities are costly and high so that it suffer from limited application in real time as introduced by [Pisarenko (1973)].

Pisarenko while his exam to frequency estimation of complex signals within white Gaussian noise, he found frequency could be derived from the eigenvector corresponding to the minimum eigenvalue of autocorrelation matrix as by [Liu et al (2011)]. Lanczos algorithm used where the eigenvalue problem for matrices of very large order are commonly found by [Biswa (1994)].

In this paper, the problem of estimating frequencies of received signal is addressed based on Pisarenko method as it does not involve eigenvalue decomposition (EVD) or singular value decomposition (SVD) of the covariance matrix of the received signals. The performance of the Pisarenko method and the proposed method is illustrated through MATLAB simulations.

**2. Pisarenko Harmonic Decomposition:**

Pisarenko harmonic decomposition, also referred to as Pisarenko's method, is a method of [frequency estimation](http://en.wikipedia.org/wiki/Frequency_estimation) as given by Hayes and Monson (1996). The auto-regressive moving average ARMA (pole-zero) model assumes that a time series *Xn* can be modeled as the output excited by a white noise (*nn*) as mentioned by [Ahmad, Schlindwein and André (2010)] i.e.:

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|  | (1) |

Where: *p*, *q* are model orders for the AR and the MA parts, respectively *a*, *b* are model parameter (*ak* being the *k*th parameter of the *p*th order model). Pisarenko assumed the stochastic process to be consisting solely of sinusoids in additive white noise, these sinusoids are assumed, in general, to be non-harmonically related. A *2pth*-orderdifference equation of real coefficients of the form:

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|  | (2) |

Can represent a deterministic process consisting of *p* real sinusoids of the form *sin()* such that *-1/2∆t ≤ fi ≤ 1/2∆t,* in this case the (*am*) are coefficients of the polynomial:



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If *Wn ,* represent additive white noise, then,

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|  | (3) |

Will represent the observed noisy process, Hence;

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|  | (4) |

Where is the noise power and since the noise is assumed to be uncorrelated with the sinusoids, then as give by [Kumaresan and Shaw (1985)]:



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|  | (5) |

Substituting *Xn-m=Yn-m – Wn-m* into equation (3), the equation can be rewritten as:

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|  | (6) |

Where a0=1, Equation (6) has the structure of ARMA with special symmetry in which the AR parameters are identical to the parameters of the MA portion of the model. The equivalent matrix expression of equation (6) is as follows:

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|  | (7) |

Multiplying both side of equation (7) by *Y*, substituting *Yn=Xn + Wn* and tacking the expectation gives:

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| --- | --- |
|  | (8) |

But

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| --- | --- |
|  | (9) |

And

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| --- | --- |
|  | (10) |

Where:

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| *RYY*  is the autocorrelation matrix.  is the noise variance.  I is the identity matrix. |

From equations (8) (9) and (10) as by Liu et al (2011):

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| --- | --- |
|  | (11) |

Equation (11) forms the basis of harmonic decomposition procedure developed by Pisarenko as mentioned by [Kay (1988) and Kumaresan and Shaw (2011)], which gives the exact frequencies and powers of real signal in white noise. The method involves the solution of an equation for determining the values of the parameters of an auto regression moving average process from the values of the autocorrelation function which constitutes the natural equation of the process, in which the variance of the noise σ2*w* is the eigenvalue of the autocorrelation matrix *RYY*. The vector of auto regression moving average parameters *A* is an eigenvector related to the eigenvalue of σ2*w*. The existence of a priori information on variance of the noise σ2*w* is, moreover, not a mandatory condition. It may be shown that for a process that consists of *p* real sinusoids and additive white noise, the variance σ2*w* corresponds to the minimal eigenvalue of the matrix *RYY .*

If the number of sinusoids is not known whereas the values of the autocorrelation function are known precisely, then independently of the solution, Equation (11) for successively growing orders or values of the auto regression moving average parameters must be calculated until the minimal eigenvalue no longer varies in a transition to the next higher order, which will also indicate that a correct order has been attained. At this point, the minimal eigenvalue of the function is equal to the variance of the noise. Such a feature of the algorithm makes it possible to apply the method to realize automatic determination of the order of the auto regression model with the use of the method, it becomes possible to effectively perform an estimation of the frequencies and powers of the sinusoid. It would be useful to combine the use of the Pisarenko harmonic decomposition method with other method. For example, the Pisarenko method may be used at the stage of calculating the auto regression coefficients as a way of automatically determining the order of the model that describes the process.

**3. Subspace method without eigen decomposition:**

Most of the traditional methods have used eigenvalue to estimate signal or noise subspace depend on the covariance matrix. From the preceding discussion, the minimum eigenvalue and associated eigenvector of equation (11) must be used to give the exact frequencies of sinusoids in white noise. These decomposition methods tend to be computationally intensive as by [Xu and Kailath,(1994)]. Therefore A Pisarenko method based fast subspace decomposition using Lanczos algorithm is proposed to estimate the exact frequencies of sinusoids in white noise. The Lanczos algorithm shown by [Gen and Charles(1983)]. used for symmetric matrix *A∈Rm×m ,* generates a symmetric banded block tridiagonal matrix *T* having the same eigenvalues of *A*:

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|  | (12) |

Where *M* and *B* is the upper triangular, *T* is such that as by [Liu et al (2011)]:

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|  | (13) |

Where *Q* is an orthonormal matrix, and its columns are called Lanczos vector, which form a set of orthonormal basis and is asymptotically equivalent estimation of the true signal subspace [Xu and Kailath,(1994)]. The eigenvalues of *T* matrix can be estimated by [Liu et al (2011)] as:

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| --- | --- |
|  | (14) |

Where is the *j-th* column of *T.* Thus the frequency of the received data matrix can be estimated.



**4. Simulation Results:**

The main motivation of simulation is to demonstrate the performance of using Lanczos algorithm for improving the resolution of frequency estimation of the proposed method and compare with pisarenko method.

Simulation example consists of two sinusoidal signals with amplitude equal one added to Additive White Gaussian Noise AWGN for different SNR and independent trials. The mean square error is described as by [Liao, Liu, and Li (2006)] as:



The effect of snapshot variation is illustrated in figure (1) where SNR equal 15 dB the number of snapshot is changed from 30 to 390. It is clear that the proposed method is better than pisarenko method, to obtain -40 dB in frequency estimation the proposed method use 30 snapshots only while pisarenko needs 80 snapshots. Figure (2) shows the effect of signal to noise ratio variation on the capabilities of methods to estimate the frequency exactly. The SNR changed from -20 to 20 dB step 5dB in each case the proposed method have less mean square error. Figure (3) shows the effect of frequency spacing. The first normalized frequency is assumed to be 0.05 at constant 15 dB SNR while the second normalized frequency changing from 0.2 to 0.95 here it is obvious that the proposed method gives lower mean square error. Figure (4), illustrates the effect of number of sources on the estimation of frequency. It is assumed non-coherent sources available from one source to eight. It is clear that the proposed method less sensitive to the increase of number of sources.

**5. Conclusion:**

In pisarenko harmonic decomposition method, the determination of the unknown frequency by checking the minimum eigenvalue requires several iteration fort the solution of the equation (11) Which is not often clear, because the estimated auto correlation lags rather than the true lags in used, moreover it consumes more computational time (i.e. computationally expensive), the proposed method uses Lanczos algorithm instead of eigen decomposition to estimate the subspace. The proposed method shows an improved performance over pisarenko method. The frequencies are estimated by eigen decomposition of the covariance matrix for the received data while the proposed method can be simply concluded as follows:

1. Transformed the covariance matrix in to tridiagonal matrix *T* using Lanczos method.
2. From *T* matrix estimated eigenvalues and corresponding eigenvector is calculated.

The proposed method is proved to be very useful, it just needs less snapshot to estimate the frequencies.

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| Fig.(1): snapshot versus mean square error  of frequency estimation | Fig.(2): Signal to noise ratio versus mean  square error of frequency estimation |
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| Fig.(3): normalized frequency spacing versus  mean square error of frequency estimation | Fig.(4): number of sources versus mean square  error of frequency estimation |