1- Constitutive Relations of Heat Flux

By the second law of thermodynamics, there exists a physical quantity \( Q \) that is, at a given time instant, associated with each surface in a non-isothermal body. This quantity can be interpreted as the heat through the surface and has two fundamental properties: behaving additively on compatible material surfaces and satisfying the first law of thermodynamics (the conservation of energy). These two properties, when rendered precisely, imply the existence of the flux vector field \( \mathbf{q} \) whose scalar product with the unit normal vector to the surface yields the surface density of the heat \( Q \) (Šilhavy 1985). \( \mathbf{q} \) is therefore named the heat-flux density vector, or the heat flux for short. The relation between the heat flux \( \mathbf{q} \) and the temperature gradient \( \nabla T \) is called the constitutive relation of heat flux, or the constitutive relation for short. It is the most fundamental and important relation in heat conduction, and is normally given by fundamental laws.

1-1 The Fourier Law

In deriving the classical heat-conduction equation, we have used the Fourier law of heat conduction. It was the first constitutive relation of heat flux and was proposed by the French mathematical physicist Joseph Fourier in 1807 based on experimentation and investigation (Wang 1994). For heat conduction in a homogeneous and isotropic medium, the Fourier law of heat conduction reads

\[
\mathbf{q}(\mathbf{r}, t) = -k \nabla T(\mathbf{r}, t) \tag{1.1}
\]

where \( \mathbf{r} \) stands for the material point, \( t \) the time, \( T \) the temperature and \( \nabla \) the gradient operator. \( k \) is the thermal conductivity of the material, which is a thermodynamic state property. By the state
theorem of thermodynamics, \( k \) should be a function of two independent and intensive dynamic properties (normally pressure and temperature; Cengel and Boles 2006). The second law of thermodynamics requires that \( k \) is positive-definite (Wang 1994, 1995, 2001). In engineering applications, we often take \( k \) as a material constant because variations in pressure and temperature are normally sufficiently small. The value of \( k \) is material-dependent. If the material is not homogeneous or isotropic, \( k \) becomes a second order tensor (Wang 1994, 1995,1996, 2001). Along with the first law of thermodynamics, this equation gives the classical *parabolic* heat-conduction equation

\[
\frac{1}{\alpha} \frac{\partial T}{\partial t} = \Delta T + \frac{1}{k} F. \tag{1.2}
\]

Here \( \alpha \) is the thermal diffusivity of the material, \( F \) is the rate of internal energy generation per unit volume, and \( \Delta \) is the Laplacian. The Fourier law of heat conduction is an early empirical law. It assumes that \( q \) and \( \nabla T \) appear at the same time instant \( t \) and consequently implies that thermal signals propagate with an infinite speed. If the material is subjected to a thermal disturbance, the effects of the disturbance will be felt instantaneously at distances infinitely far from its source. Although this result is physically unrealistic, it has been confirmed by many experiments that the Fourier law of heat conduction holds for many media in the usual range of heat flux \( q \) and temperature gradient (Wang1994).
1-2 The CV Constitutive Relation

With the development of science and technology such as the application of ultrafast pulse-laser heating on metal films, heat conduction appears in the range of high heat flux and high unsteadiness. The drawback of infinite heat propagation speed in the Fourier law becomes unacceptable. This has inspired the work of searching for new constitutive relations. Among many proposed relations (Wang 1994), the constitutive relation proposed by Cattaneo (1958) and Vernotte (1958, 1961),

$$q(r, t) + \tau_0 \frac{\partial q(r, t)}{\partial t} = -k \nabla T(r, t) \quad (1.3)$$

is the most widely accepted. This relation is named the CV constitutive relation after the names of the proposers. Here $\tau_0 > 0$ is a material property and is called the relaxation time. The corresponding heat-conduction equation is thus

$$\frac{1}{\alpha} \frac{\partial T}{\partial t} + \tau_0 \frac{\partial^2 T}{\partial t^2} = \Delta T + \frac{1}{k} \left( F + \tau_0 \frac{\partial F}{\partial t} \right) \quad (1.4)$$

Unlike its classical counterpart Eq. (1.2), this equation is of hyperbolic type, characterizes the combined diffusion and wave-like behavior of heat conduction, and predicts a finite speed

$$V_{CV} = \sqrt{\frac{k}{\rho c \tau_0}} \quad (1.5)$$

Note that the CV constitutive relation is actually a first-order approximation of a more general constitutive relation (single-phase-lagging model; Tzou 1992a),

\[ q(r, t + \tau_0) = -k \nabla T(r, t). \]  

According to which the temperature gradient established at a point \( r \) at time \( t \) gives rise to a heat flux vector at \( r \) at a later time \( t + \tau_0 \). There is a finite built-up time \( \tau_0 \) for the onset of heat flux at \( r \) after a temperature gradient is imposed there. Thus the \( \tau_0 \) represents the time lag needed to establish the heat flux (the result) when a temperature gradient (the cause) is suddenly imposed. The higher \( \partial q/\partial t \) corresponds to a larger derivation of the CV constitutive relation from the classical Fourier law.

The value of \( \tau_0 \) is material-dependent (Chandrasekharaiha 1986, 1998, Tzou1997). For most solid materials, \( \tau_0 \) varies from \( 10^{-10} \) s to \( 10^{-14} \) s. For gases, \( \tau_0 \) is normally in the range of \( 10^{-8} \sim 10^{-10} \) s. The value of \( \tau_0 \) for some biological materials and materials with non-homogeneous inner structures can be up to \( 10^2 \) s (Beckert 2000, Kaminski 1990, Mitra et al. 1995, Peters 1999, Roetzel et al. 2003, Vedavarz et al. 1992). Therefore, the thermal relaxation effects can be of relevance even in common engineering applications where the time scales of interest are of the order of a fraction of a minute.

Three factors contribute to the significance of the second term in the hyperbolic heat-conduction equation (1.4): the value of \( \tau_0 \), the rate of change of temperature, and the time scale involved. The
wave nature of thermal signals will be over the diffusive behavior through this term when (Tzou 1992a)

\[ \frac{\partial T}{\partial t} \gg \frac{T_r}{2\tau_0} \exp(t/\tau_0) \]  

(1.7)

where \( T_r \) is a reference temperature. Therefore, the wave-like features will become significant when: (1) \( \tau_0 \) is large, (2) \( \partial T/\partial t \) is high, or (3) \( t \) is small. Some typical situations where hyperbolic heat conduction differs from classical parabolic heat conduction include those concerned with a localized moving heat source with high intensity, a rapidly propagating crack tip, shock wave propagation, thermal resonance, interfacial effects between dissimilar materials, laser material processing, and laser surgery (Chandrasekharahiah 1986, 1998, Joseph and Preziosi 1989, 1990, Tzou 1992a, 1995a, 1997, Wang 1994, 2000a).

When \( \tau_0 \rightarrow \infty \) but \( k_{eff} = k/\tau_0 \) is finite, the CV constitutive relation (3) and the hyperbolic heat-conduction equation (4) become (Joseph and Preziosi 1989)

\[ \frac{\partial q(r,t)}{\partial t} = -k_{eff} \nabla T(r,t) \]  

(1.8)

And

\[ \frac{1}{\alpha_{eff}} \frac{\partial^2 T}{\partial t^2} = \Delta T + \frac{1}{k_{eff}} \frac{\partial F}{\partial t} \]  

(1.9)

where \( \alpha_{eff} = k_{eff}/\rho c \), \( \rho \) and \( c \) are the density and the specific heat of the material, respectively. Therefore, when \( \tau_0 \) is very large, a temperature gradient established at a point of the material results in an instantaneous heat flux rate at that point, and vice-versa. Eq. (1.9) is a classical wave equation that predicts thermal wave
propagation with speed $VCV$, like Eq. (1.4). A major difference exists, however, between Eqs. (1.4) and (1.9): the former allows damping of thermal waves, the latter does not (Wang and Zhou 2000, 2001).

2- Three Types of Heat-Conduction Equations

Heat-conduction equations come from the application of the first law of thermodynamics (also called the conservation of energy) to heat conduction. and the formula for calculating internal energy in thermodynamics, the first law of thermodynamics yields

$$\rho c \frac{\partial T(r, t)}{\partial t} + \nabla \cdot q - \Gamma(r, t) = 0 \quad (2.1)$$

where $\rho$ and $c$ are the density and the specific heat of the material, and $\Gamma$ is the rate of internal energy generation per unit volume. Equation (2.1) is called the energy equation, and it contains two unknowns $T$ and $q$. By using a constitutive relation of heat flux density, we may eliminate $q$ from Eq. (2.1) to obtain an equation of temperature $T$.

Assuming constant material properties,

$$\nabla \cdot q + \tau_0 \frac{\partial}{\partial t} [\nabla \cdot q] = -k \Delta T - k \tau_T \frac{\partial}{\partial t} [\Delta T] \quad (2.2)$$

Substituting the expression of $\nabla \cdot q$ from Eq. (2.2)

$$\nabla \cdot q = F - \rho c \frac{\partial T}{\partial t}$$

into Eq. (2.2) and introducing the thermal diffusivity $\alpha = k/(\rho c)$ leads to
This is called the dual-phase-lagging heat-conduction equation. When \( \tau_T = 0 \), it reduces to the hyperbolic heat-conduction equation

\[
\frac{1}{\alpha} \frac{\partial T}{\partial t} + \frac{\tau_0}{\alpha} \frac{\partial^2 T}{\partial t^2} = \Delta T + \tau_T \frac{\partial}{\partial t} (\Delta T) + \frac{1}{k} \left( F + \tau_0 \frac{\partial F}{\partial t} \right)
\]  

(2.3)

In the absence of two phase lags, i.e. when \( \tau_0 = \tau_T = 0 \), it reduces to the classical parabolic heat-conduction equation

\[
\frac{1}{\alpha} \frac{\partial T}{\partial t} = \Delta T + \frac{1}{k} F.
\]  

(2.5)

For steady-state heat conduction, both the first and the second derivatives of \( T \) with respect to \( t \) are zero. All three kinds of heat-conduction equations reduce to potential equations.