EXAMPLE 1 Derivatives Involving the Sine

(a) $y = x^2 - \sin x$:

$$\frac{dy}{dx} = 2x - \frac{d}{dx}(\sin x) \qquad \text{Difference Rule}$$
$$= 2x - \cos x.$$

(b) $y = x^2 \sin x$:

$$\frac{dy}{dx} = x^2 \frac{d}{dx}(\sin x) + 2x \sin x \qquad \text{Product Rule}$$
$$= x^2 \cos x + 2x \sin x.$$

(c) $y = \dfrac{\sin x}{x}$:

$$\frac{dy}{dx} = \frac{x \cdot \frac{d}{dx}(\sin x) - \sin x \cdot 1}{x^2} \qquad \text{Quotient Rule}$$
$$= \frac{x \cos x - \sin x}{x^2}.$$

EXAMPLE 2 Derivatives Involving the Cosine

(a) $y = 5x + \cos x$:

$$\frac{dy}{dx} = \frac{d}{dx}(5x) + \frac{d}{dx}(\cos x) \qquad \text{Sum Rule}$$
$$= 5 - \sin x.$$

(b) $y = \sin x \cos x$:

$$\frac{dy}{dx} = \sin x \frac{d}{dx}(\cos x) + \cos x \frac{d}{dx}(\sin x) \qquad \text{Product Rule}$$
$$= \sin x(-\sin x) + \cos x(\cos x)$$
$$= \cos^2 x - \sin^2 x.$$

(c) $y = \dfrac{\cos x}{1 - \sin x}$:

$$\frac{dy}{dx} = \frac{(1 - \sin x)\frac{d}{dx}(\cos x) - \cos x \frac{d}{dx}(1 - \sin x)}{(1 - \sin x)^2} \qquad \text{Quotient Rule}$$
$$= \frac{(1 - \sin x)(-\sin x) - \cos x(0 - \cos x)}{(1 - \sin x)^2}$$
$$= \frac{1 - \sin x}{(1 - \sin x)^2} \qquad \sin^2 x + \cos^2 x = 1$$
$$= \frac{1}{1 - \sin x}.$$

EX: Prove that

$$\frac{d}{dx}\left(\tan x\right) = \sec^2 x$$

Proof :

$$\frac{d}{dx}\left(\tan x\right) = \frac{d}{dx}\left(\frac{\sin x}{\cos x}\right) = \frac{\cos x \frac{d}{dx}\left(\sin x\right) - \sin x \frac{d}{dx}\left(\cos x\right)}{\cos^2 x} \qquad \text{Quotient Rule}$$

$$= \frac{\cos x \cos x - \sin x\left(-\sin x\right)}{\cos^2 x}$$

$$= \frac{\cos^2 x + \sin^2 x}{\cos^2 x}$$

$$= \frac{1}{\cos^2 x} = \sec^2 x$$

EX : Find :

Find y'' if $y = \sec x$

SOL :

$$y = \sec x$$

$$y' = \sec x \tan x$$

$$y'' = \frac{d}{dx}(\sec x \tan x)$$

$$= \sec x \frac{d}{dx}\left(\tan x\right) + \tan x \frac{d}{dx}\left(\sec x\right) \qquad \text{Product Rule}$$

$$= \sec x(\sec^2 x) + \tan x(\sec x \tan x)$$

$$= \sec^3 x + \sec x \tan^2 x$$

EX :

Finding a Trigonometric Limit

$$\lim_{x \to 0} \frac{\sqrt{2 + \sec x}}{\cos(\pi - \tan x)}$$

SOL :

$$\lim_{x \to 0} \frac{\sqrt{2 + \sec x}}{\cos(\pi - \tan x)} = \frac{\sqrt{2 + \sec 0}}{\cos(\pi - \tan 0)} = \frac{\sqrt{2 + 1}}{\cos(\pi - 0)} = \frac{\sqrt{3}}{-1} = -\sqrt{3}$$

EX : Find $\frac{dy}{dx}$:

1. $y = -10x + 3\cos x$

2. $y = \frac{3}{x} + 5\sin x$

3. $y = \csc x - 4\sqrt{x} + 7$

4. $y = x^2 \cot x - \frac{1}{x^2}$

5. $y = (\sec x + \tan x)(\sec x - \tan x)$

6. $y = (\sin x + \cos x)\sec x$

7. $y = \frac{\cot x}{1 + \cot x}$

8. $y = \frac{\cos x}{1 + \sin x}$

9. $y = \frac{4}{\cos x} + \frac{1}{\tan x}$

10. $y = \frac{\cos x}{x} + \frac{x}{\cos x}$

11. $y = x^2 \sin x + 2x\cos x - 2\sin x$

12. $y = x^2 \cos x - 2x\sin x - 2\cos x$

SOLUTIONS :

1. $y = -10x + 3\cos x \Rightarrow \frac{dy}{dx} = -10 + 3\frac{d}{dx}(\cos x) = -10 - 3\sin x$

2. $y = \frac{3}{x} + 5\sin x \Rightarrow \frac{dy}{dx} = \frac{-3}{x^2} + 5\frac{d}{dx}(\sin x) = \frac{-3}{x^2} + 5\cos x$

3. $y = \csc x - 4\sqrt{x} + 7 \Rightarrow \frac{dy}{dx} = -\csc x \cot x - \frac{4}{2\sqrt{x}} + 0 = -\csc x \cot x - \frac{2}{\sqrt{x}}$

4. $y = x^2 \cot x - \frac{1}{x^2} \Rightarrow \frac{dy}{dx} = x^2 \frac{d}{dx}(\cot x) + \cot x \cdot \frac{d}{dx}(x^2) + \frac{2}{x^3} = -x^2 \csc^2 x + (\cot x)(2x) + \frac{2}{x^3}$
$= -x^2 \csc^2 x + 2x\cot x + \frac{2}{x^3}$

5. $y = (\sec x + \tan x)(\sec x - \tan x) \Rightarrow \frac{dy}{dx} = (\sec x + \tan x)\frac{d}{dx}(\sec x - \tan x) + (\sec x - \tan x)\frac{d}{dx}(\sec x + \tan x)$
$= (\sec x + \tan x)(\sec x \tan x - \sec^2 x) + (\sec x - \tan x)(\sec x \tan x + \sec^2 x)$
$= (\sec^2 x \tan x + \sec x \tan^2 x - \sec^3 x - \sec^2 x \tan x) + (\sec^2 x \tan x - \sec x \tan^2 x + \sec^3 x - \tan x \sec^2 x) = 0.$
$\left(\text{Note also that } y = \sec^2 x - \tan^2 x = (\tan^2 x + 1) - \tan^2 x = 1 \Rightarrow \frac{dy}{dx} = 0.\right)$

6. $y = (\sin x + \cos x)\sec x \Rightarrow \frac{dy}{dx} = (\sin x + \cos x)\frac{d}{dx}(\sec x) + \sec x \frac{d}{dx}(\sin x + \cos x)$
$= (\sin x + \cos x)(\sec x \tan x) + (\sec x)(\cos x - \sin x) = \frac{(\sin x + \cos x)\sin x}{\cos^2 x} + \frac{\cos x - \sin x}{\cos x}$
$= \frac{\sin^2 x + \cos x \sin x + \cos^2 x - \cos x \sin x}{\cos^2 x} = \frac{1}{\cos^2 x} = \sec^2 x$
$\left(\text{Note also that } y = \sin x \sec x + \cos x \sec x = \tan x + 1 \Rightarrow \frac{dy}{dx} = \sec^2 x.\right)$

7. $y = \frac{\cot x}{1+\cot x} \Rightarrow \frac{dy}{dx} = \frac{(1+\cot x)\frac{d}{dx}(\cot x) - (\cot x)\frac{d}{dx}(1+\cot x)}{(1+\cot x)^2} = \frac{(1+\cot x)(-\csc^2 x) - (\cot x)(-\csc^2 x)}{(1+\cot x)^2}$
$= \frac{-\csc^2 x - \csc^2 x \cot x + \csc^2 x \cot x}{(1+\cot x)^2} = \frac{-\csc^2 x}{(1+\cot x)^2}$

8. $y = \frac{\cos x}{1+\sin x} \Rightarrow \frac{dy}{dx} = \frac{(1+\sin x)\frac{d}{dx}(\cos x) - (\cos x)\frac{d}{dx}(1+\sin x)}{(1+\sin x)^2} = \frac{(1+\sin x)(-\sin x) - (\cos x)(\cos x)}{(1+\sin x)^2}$
$= \frac{-\sin x - \sin^2 x - \cos^2 x}{(1+\sin x)^2} = \frac{-\sin x - 1}{(1+\sin x)^2} = \frac{-(1+\sin x)}{(1+\sin x)^2} = \frac{-1}{1+\sin x}$

9. $y = \frac{4}{\cos x} + \frac{1}{\tan x} = 4\sec x + \cot x \Rightarrow \frac{dy}{dx} = 4\sec x \tan x - \csc^2 x$

10. $y = \frac{\cos x}{x} + \frac{x}{\cos x} \Rightarrow \frac{dy}{dx} = \frac{x(-\sin x) - (\cos x)(1)}{x^2} + \frac{(\cos x)(1) - x(-\sin x)}{\cos^2 x} = \frac{-x\sin x - \cos x}{x^2} + \frac{\cos x + x\sin x}{\cos^2 x}$

11. $y = x^2 \sin x + 2x\cos x - 2\sin x \Rightarrow \frac{dy}{dx} = (x^2\cos x + (\sin x)(2x)) + ((2x)(-\sin x) + (\cos x)(2)) - 2\cos x$
$= x^2\cos x + 2x\sin x - 2x\sin x + 2\cos x - 2\cos x = x^2\cos x$

12. $y = x^2\cos x - 2x\sin x - 2\cos x \Rightarrow \frac{dy}{dx} = (x^2(-\sin x) + (\cos x)(2x)) - (2x\cos x + (\sin x)(2)) - 2(-\sin x)$
$= -x^2\sin x + 2x\cos x - 2x\cos x - 2\sin x + 2\sin x = -x^2\sin x$

EX : Find $\frac{ds}{dt}$:

13. $s = \tan t - t$

14. $s = t^2 - \sec t + 1$

15. $s = \frac{1 + \csc t}{1 - \csc t}$

16. $s = \frac{\sin t}{1 - \cos t}$

SOLUTIONS :

13. $s = \tan t - t \Rightarrow \frac{ds}{dt} = \frac{d}{dt}(\tan t) - 1 = \sec^2 t - 1 = \tan^2 t$

14. $s = t^2 - \sec t + 1 \Rightarrow \frac{ds}{dt} = 2t - \frac{d}{dt}(\sec t) = 2t - \sec t \tan t$

15. $s = \frac{1+\csc t}{1-\csc t} \Rightarrow \frac{ds}{dt} = \frac{(1-\csc t)(-\csc t \cot t) - (1+\csc t)(\csc t \cot t)}{(1-\csc t)^2}$
$= \frac{-\csc t \cot t + \csc^2 t \cot t - \csc t \cot t - \csc^2 t \cot t}{(1-\csc t)^2} = \frac{-2\csc t \cot t}{(1-\csc t)^2}$

16. $s = \frac{\sin t}{1-\cos t} \Rightarrow \frac{ds}{dt} = \frac{(1-\cos t)(\cos t) - (\sin t)(\sin t)}{(1-\cos t)^2} = \frac{\cos t - \cos^2 t - \sin^2 t}{(1-\cos t)^2} = \frac{\cos t - 1}{(1-\cos t)^2} = -\frac{1}{1-\cos t}$
$= \frac{1}{\cos t - 1}$

EX : Find $\frac{dr}{d\theta}$:

17. $r = 4 - \theta^2 \sin\theta$

18. $r = \theta \sin\theta + \cos\theta$

19. $r = \sec\theta \csc\theta$

20. $r = (1 + \sec\theta)\sin\theta$

SOLUTIONS :

17. $\mathrm{r} = 4 - \theta^2 \sin\theta \;\Rightarrow\; \frac{dr}{d\theta} = -\left(\theta^2 \frac{d}{d\theta}(\sin\theta) + (\sin\theta)(2\theta)\right) = -(\theta^2 \cos\theta + 2\theta \sin\theta) = -\theta(\theta\cos\theta + 2\sin\theta)$

18. $\mathrm{r} = \theta \sin\theta + \cos\theta \;\Rightarrow\; \frac{dr}{d\theta} = (\theta\cos\theta + (\sin\theta)(1)) - \sin\theta = \theta\cos\theta$

19. $\mathrm{r} = \sec\theta \csc\theta \;\Rightarrow\; \frac{dr}{d\theta} = (\sec\theta)(-\csc\theta\cot\theta) + (\csc\theta)(\sec\theta\tan\theta)$
$= \left(\frac{-1}{\cos\theta}\right)\left(\frac{1}{\sin\theta}\right)\left(\frac{\cos\theta}{\sin\theta}\right) + \left(\frac{1}{\sin\theta}\right)\left(\frac{1}{\cos\theta}\right)\left(\frac{\sin\theta}{\cos\theta}\right) = \frac{-1}{\sin^2\theta} + \frac{1}{\cos^2\theta} = \sec^2\theta - \csc^2\theta$

20. $\mathrm{r} = (1 + \sec\theta)\sin\theta \;\Rightarrow\; \frac{dr}{d\theta} = (1 + \sec\theta)\cos\theta + (\sin\theta)(\sec\theta\tan\theta) = (\cos\theta + 1) + \tan^2\theta = \cos\theta + \sec^2\theta$

EX : Find $\frac{dp}{dq}$:

21. $p = 5 + \frac{1}{\cot q}$

22. $p = (1 + \csc q)\cos q$

23. $p = \frac{\sin q + \cos q}{\cos q}$

24. $p = \frac{\tan q}{1 + \tan q}$

SOLUTIONS :

21. $\mathrm{p} = 5 + \frac{1}{\cot q} = 5 + \tan q \;\Rightarrow\; \frac{dp}{dq} = \sec^2 q$

22. $\mathrm{p} = (1 + \csc q)\cos q \;\Rightarrow\; \frac{dp}{dq} = (1 + \csc q)(-\sin q) + (\cos q)(-\csc q\cot q) = (-\sin q - 1) - \cot^2 q = -\sin q - \csc^2 q$

23. $\mathrm{p} = \frac{\sin q + \cos q}{\cos q} \;\Rightarrow\; \frac{dp}{dq} = \frac{(\cos q)(\cos q - \sin q) - (\sin q + \cos q)(-\sin q)}{\cos^2 q}$
$= \frac{\cos^2 q - \cos q \sin q + \sin^2 q + \cos q \sin q}{\cos^2 q} = \frac{1}{\cos^2 q} = \sec^2 q$

24. $\mathrm{p} = \frac{\tan q}{1 + \tan q} \;\Rightarrow\; \frac{dp}{dq} = \frac{(1 + \tan q)(\sec^2 q) - (\tan q)(\sec^2 q)}{(1 + \tan q)^2} = \frac{\sec^2 q + \tan q \sec^2 q - \tan q \sec^2 q}{(1 + \tan q)^2} = \frac{\sec^2 q}{(1 + \tan q)^2}$

EX :

25. Find y'' if

a. $y = \csc x$. **b.** $y = \sec x$.

SOLUTIONS :

25. (a) $y = \csc x \Rightarrow y' = -\csc x \cot x \Rightarrow y'' = -\left((\csc x)\left(-\csc^2 x\right) + (\cot x)(-\csc x \cot x)\right) = \csc^3 x + \csc x \cot^2 x$
$= (\csc x)\left(\csc^2 x + \cot^2 x\right) = (\csc x)\left(\csc^2 x + \csc^2 x - 1\right) = 2\csc^3 x - \csc x$

(b) $y = \sec x \Rightarrow y' = \sec x \tan x \Rightarrow y'' = (\sec x)\left(\sec^2 x\right) + (\tan x)(\sec x \tan x) = \sec^3 x + \sec x \tan^2 x$
$= (\sec x)\left(\sec^2 x + \tan^2 x\right) = (\sec x)\left(\sec^2 x + \sec^2 x - 1\right) = 2\sec^3 x - \sec x$

EX :

26. Find $y^{(4)} = d^4 y/dx^4$ if

a. $y = -2\sin x$. **b.** $y = 9\cos x$

SOLUTIONS :

26. (a) $y = -2\sin x \Rightarrow y' = -2\cos x \Rightarrow y'' = -2(-\sin x) = 2\sin x \Rightarrow y''' = 2\cos x \Rightarrow y^{(4)} = -2\sin x$

(b) $y = 9\cos x \Rightarrow y' = -9\sin x \Rightarrow y'' = -9\cos x \Rightarrow y''' = -9(-\sin x) = 9\sin x \Rightarrow y^{(4)} = 9\cos x$

EX : Find the following limits :

39. $\lim_{x\to 2} \sin\left(\frac{1}{x} - \frac{1}{2}\right)$

40. $\lim_{x\to -\pi/6} \sqrt{1 + \cos(\pi \csc x)}$

41. $\lim_{x\to 0} \sec\left[\cos x + \pi \tan\left(\frac{\pi}{4\sec x}\right) - 1\right]$

42. $\lim_{x\to 0} \sin\left(\frac{\pi + \tan x}{\tan x - 2\sec x}\right)$

43. $\lim_{t\to 0} \tan\left(1 - \frac{\sin t}{t}\right)$

44. $\lim_{\theta\to 0} \cos\left(\frac{\pi\theta}{\sin\theta}\right)$

SOLUTIONS :

39. $\lim_{x\to 2} \sin\left(\frac{1}{x} - \frac{1}{2}\right) = \sin\left(\frac{1}{2} - \frac{1}{2}\right) = \sin 0 = 0$

40. $\lim_{x\to \frac{\pi}{6}} \sqrt{1 + \cos(\pi \csc x)} = \sqrt{1 + \cos\left(\pi \csc \frac{\pi}{6}\right)} = \sqrt{1 + \cos(\pi \cdot 2)} = \sqrt{2}$

41. $\lim_{x\to 0} \sec\left[\cos x + \pi \tan\left(\frac{\pi}{4\sec x}\right) - 1\right] = \sec\left[\cos 0 + \pi \tan\left(\frac{\pi}{4\sec 0}\right) - 1\right] = \sec\left[1 + \pi \tan\left(\frac{\pi}{4}\right) - 1\right] = \sec \pi = -1$

42. $\lim\limits_{x \to 0} \sin\left(\frac{\pi+\tan x}{\tan x - 2\sec x}\right) = \sin\left(\frac{\pi+\tan 0}{\tan 0 - 2\sec 0}\right) = \sin\left(-\frac{\pi}{2}\right) = -1$

43. $\lim\limits_{t \to 0} \tan\left(1 - \frac{\sin t}{t}\right) = \tan\left(1 - \lim\limits_{t \to 0} \frac{\sin t}{t}\right) = \tan(1-1) = 0$

44. $\lim\limits_{\theta \to 0} \cos\left(\frac{\pi\theta}{\sin\theta}\right) = \cos\left(\pi \lim\limits_{\theta \to 0} \frac{\theta}{\sin\theta}\right) = \cos\left(\pi \cdot \frac{1}{\lim\limits_{\theta\to 0} \frac{\sin\theta}{\theta}}\right) = \cos\left(\pi \cdot \frac{1}{1}\right) = -1$

EX :

50. Derive the formula for the derivative with respect to x of

a. $\sec x$. **b.** $\csc x$. **c.** $\cot x$.

SOLUTIONS :

50. (a) $y = \sec x = \frac{1}{\cos x} \Rightarrow \frac{dy}{dx} = \frac{(\cos x)(0) - (1)(-\sin x)}{(\cos x)^2} = \frac{\sin x}{\cos^2 x} = \left(\frac{1}{\cos x}\right)\left(\frac{\sin x}{\cos x}\right) = \sec x \tan x$
$\Rightarrow \frac{d}{dx}(\sec x) = \sec x \tan x$

(b) $y = \csc x = \frac{1}{\sin x} \Rightarrow \frac{dy}{dx} = \frac{(\sin x)(0) - (1)(\cos x)}{(\sin x)^2} = \frac{-\cos x}{\sin^2 x} = \left(\frac{-1}{\sin x}\right)\left(\frac{\cos x}{\sin x}\right) = -\csc x \cot x$
$\Rightarrow \frac{d}{dx}(\csc x) = -\csc x \cot x$

(c) $y = \cot x = \frac{\cos x}{\sin x} \Rightarrow \frac{dy}{dx} = \frac{(\sin x)(-\sin x) - (\cos x)(\cos x)}{(\sin x)^2} = \frac{-\sin^2 x - \cos^2 x}{\sin^2 x} = \frac{-1}{\sin^2 x} = -\csc^2 x$
$\Rightarrow \frac{d}{dx}(\cot x) = -\csc^2 x$