Method [4]

Integration By Parts

The formula :

$$\int u\,dv = uv - \int v\,du \qquad (2)$$

Consider

$$w = u.v \quad \Rightarrow \quad dw = u.dv + +v.du \quad \Rightarrow \quad u\,dv = dw - v\,du$$

$$\int u\,dv = \int dw - \int v\,du = w - \int v\,du$$

$$\boxed{\int u\ dv = u\,.v - \int v\,.du}$$

Evaluate $I = \int \ln(x)\,dx$

Solution :-

$$u = \ln(x) \qquad dv = dx$$

$$du = \frac{dx}{x} \qquad v = x$$

$$\Rightarrow I = x\ln(x) - \int x\frac{1}{x}dx = x\ln(x) - \int dx = x\ln(x) - x + c$$

Evaluate $I = \int \tan^{-1}(x)dx$

Solution :-

$$u = \tan^{-1}(x) \qquad dv = dx$$

$$du = \frac{dx}{1+x^2} \qquad v = x$$

$$\Rightarrow \quad I = x\tan^{-1}(x) - \int \frac{xdx}{1+x^2} = x\tan^{-1}(x) - \frac{1}{2}\ln(1+x^2) + c$$

Evaluate $I = \int xe^x dx$

Solution :-

$$u = x \qquad dv = e^x dx$$

$$du = dx \qquad v = e^x$$

$$\Rightarrow \quad I = xe^x - \int e^x\,dx = xe^x - e^x + c$$

Evaluate $I = \int e^x \sin(x)dx$

Solution :-

$$u = e^x \qquad dv = \sin(x)dx$$
$$du = e^x dx \qquad v = -\cos(x)$$
$$I = -e^x \cos(x) + \int e^x \cos(x)dx = -e^x \cos(x) + j$$

Where $\quad j = \int e^x \cos(x)dx \Rightarrow \quad u = e^x \qquad dv = \cos(x)dx$

$$du = e^x dx \qquad v = \sin(x)$$

$$j = e^x \sin(x) - \int e^x \sin(x)dx = e^x \sin(x) - I$$

$$I = -e^x \cos(x) + e^x \sin(x) - I \quad \Rightarrow \quad 2I = -e^x \cos(x) + e^x \sin(x)$$
$$\Rightarrow I = \frac{1}{2} e^x (\sin(x) - \cos(x))$$

EX : Evaluate the following integrals by using integration by parts .

1. $\int x \sin\frac{x}{2} dx \qquad$ **2.** $\int \theta \cos \pi\theta \, d\theta$

5. $\int_1^2 x \ln x \, dx \qquad$ **6.** $\int_1^e x^3 \ln x \, dx$

SOL :

1. $u = x,\ du = dx;\ dv = \sin\frac{x}{2}\, dx,\ v = -2\cos\frac{x}{2}$;
$$\int x \sin\frac{x}{2}\, dx = -2x\cos\frac{x}{2} - \int\left(-2\cos\frac{x}{2}\right) dx = -2x\cos\left(\frac{x}{2}\right) + 4\sin\left(\frac{x}{2}\right) + C$$

2. $u = \theta,\ du = d\theta;\ dv = \cos\pi\theta\, d\theta,\ v = \frac{1}{\pi}\sin\pi\theta$;
$$\int \theta\cos\pi\theta\, d\theta = \frac{\theta}{\pi}\sin\pi\theta - \int\frac{1}{\pi}\sin\pi\theta\, d\theta = \frac{\theta}{\pi}\sin\pi\theta + \frac{1}{\pi^2}\cos\pi\theta + C$$

5. $u = \ln x,\ du = \frac{dx}{x};\ dv = x\, dx,\ v = \frac{x^2}{2}$;
$$\int_1^2 x\ln x\, dx = \left[\frac{x^2}{2}\ln x\right]_1^2 - \int_1^2 \frac{x^2}{2}\,\frac{dx}{x} = 2\ln 2 - \left[\frac{x^2}{4}\right]_1^2 = 2\ln 2 - \frac{3}{4} = \ln 4 - \frac{3}{4}$$

6. $u = \ln x,\ du = \frac{dx}{x};\ dv = x^3\, dx,\ v = \frac{x^4}{4}$;
$$\int_1^e x^3\ln x\, dx = \left[\frac{x^4}{4}\ln x\right]_1^e - \int_1^e \frac{x^4}{4}\,\frac{dx}{x} = \frac{e^4}{4} - \left[\frac{x^4}{16}\right]_1^e = \frac{3e^4+1}{16}$$

EX : Evaluate the following integral by using integration by parts .

$$\int e^x \cos x\, dx$$

SOL :

Let $u = e^x$ and $dv = \cos x\, dx$. Then $du = e^x\, dx$, $v = \sin x$, and

$$\int e^x \cos x\, dx = e^x \sin x - \int e^x \sin x\, dx.$$

The second integral is like the first except that it has $\sin x$ in place of $\cos x$. To evaluate it, we use integration by parts with

$$u = e^x, \qquad dv = \sin x\, dx, \qquad v = -\cos x, \qquad du = e^x\, dx.$$

Then

$$\int e^x \cos x\, dx = e^x \sin x - \left(-e^x \cos x - \int (-\cos x)(e^x\, dx)\right)$$

$$= e^x \sin x + e^x \cos x - \int e^x \cos x\, dx.$$

$$2\int e^x \cos x\, dx = e^x \sin x + e^x \cos x + C$$

$$\int e^x \cos x\, dx = \frac{e^x \sin x + e^x \cos x}{2} + C$$

EX :

Find the area of the region bounded by the curve $y = xe^{-x}$ and the x-axis from $x = 0$ to $x = 4$.

SOL : The area above is finding by solving the following integration :

$$\int_0^4 xe^{-x}\, dx$$

Let $u = x$, $dv = e^{-x}\, dx$, $v = -e^{-x}$, and $du = dx$. Then,

$$\int_0^4 xe^{-x}\, dx = -xe^{-x}\Big]_0^4 - \int_0^4 (-e^{-x})\, dx$$

$$= [-4e^{-4} - (0)] + \int_0^4 e^{-x}\, dx$$

$$= -4e^{-4} - e^{-x}\Big]_0^4$$

$$= -4e^{-4} - e^{-4} - (-e^0) = 1 - 5e^{-4} \approx 0.91$$

EX : Evaluate the following integrals by using integration by parts :

1) $\int \tan^{-1} y \; dy$

SOL :

$u = \tan^{-1} y,\ du = \frac{dy}{1+y^2}\ ;\ dv = dy,\ v = y;$

$\int \tan^{-1} y\, dy = y \tan^{-1} y - \int \frac{y\, dy}{(1+y^2)} = y \tan^{-1} y - \frac{1}{2} \ln(1+y^2) + C = y \tan^{-1} y - \ln \sqrt{1+y^2} + C$

2) $\int \sin^{-1} y \; dy$

SOL :

$u = \sin^{-1} y,\ du = \frac{dy}{\sqrt{1-y^2}}\ ;\ dv = dy,\ v = y;$

$\int \sin^{-1} y\, dy = y \sin^{-1} y - \int \frac{y\, dy}{\sqrt{1-y^2}} = y \sin^{-1} y + \sqrt{1-y^2} + C$

3) $\int x \sec^2 x \; dx$

SOL :

$u = x,\ du = dx;\ dv = \sec^2 x\, dx,\ v = \tan x;$

$\int x \sec^2 x\, dx = x \tan x - \int \tan x\, dx = x \tan x + \ln|\cos x| + C$

4) $\int 4x \sec^2 2x \; dx$

SOL :

$\int 4x \sec^2 2x\, dx;\ [y = 2x]\ \rightarrow\ \int y \sec^2 y\, dy = y \tan y - \int \tan y\, dy = y \tan y - \ln|\sec y| + C$

$= 2x \tan 2x - \ln|\sec 2x| + C$