EXAMPLES :

Solve by using substitution rule :

1) $\int \sqrt{1+y^2} \cdot 2y\, dy$

$$\int \sqrt{1+y^2} \cdot 2y\, dy = \int \sqrt{u} \cdot \left(\frac{du}{dy}\right) dy$$ Let $u = 1 + y^2$, $du/dy = 2y$

$$= \int u^{1/2}\, du$$

$$= \frac{u^{(1/2)+1}}{(1/2)+1} + C$$ Integrate, using Eq. (1) with $n = 1/2$.

$$= \frac{2}{3} u^{3/2} + C$$ Simpler form

$$= \frac{2}{3}(1+y^2)^{3/2} + C$$ Replace u by $1 + y^2$.

2) $\int \sqrt{4t-1}\, dt$

$$\int \sqrt{4t-1}\, dt = \int \frac{1}{4} \cdot \sqrt{4t-1} \cdot 4\, dt$$

$$= \frac{1}{4} \int \sqrt{u} \cdot \left(\frac{du}{dt}\right) dt$$ Let $u = 4t - 1$, $du/dt = 4$.

$$= \frac{1}{4} \int u^{1/2}\, du$$ With the $1/4$ out front, the integral is now in standard form.

$$= \frac{1}{4} \cdot \frac{u^{3/2}}{3/2} + C$$ Integrate, using Eq. (1) with $n = 1/2$.

$$= \frac{1}{6} u^{3/2} + C$$ Simpler form

$$= \frac{1}{6}(4t-1)^{3/2} + C$$ Replace u by $4t - 1$.

3)

$$\int \cos(7\theta + 5)\, d\theta = \int \cos u \cdot \frac{1}{7}\, du$$

$$= \frac{1}{7} \int \cos u\, du$$

$$= \frac{1}{7} \sin u + C$$

$$= \frac{1}{7} \sin(7\theta + 5) + C$$

4)

$$\int x^2 \sin(x^3)\,dx = \int \sin(x^3)\cdot x^2\,dx$$

$$= \int \sin u \cdot \frac{1}{3}\,du$$ Let $u = x^3$, $du = 3x^2\,dx$, $(1/3)\,du = x^2\,dx$.

$$= \frac{1}{3}\int \sin u\,du$$

$$= \frac{1}{3}(-\cos u) + C$$ Integrate with respect to u.

$$= -\frac{1}{3}\cos(x^3) + C$$ Replace u by x^3.

5)

$$\int \frac{1}{\cos^2 2x}\,dx = \int \sec^2 2x\,dx$$ $\frac{1}{\cos 2x} = \sec 2x$

$$= \int \sec^2 u \cdot \frac{1}{2}\,du$$ $u = 2x$, $du = 2\,dx$, $dx = (1/2)\,du$

$$= \frac{1}{2}\int \sec^2 u\,du$$

$$= \frac{1}{2}\tan u + C$$ $\frac{d}{du}\tan u = \sec^2 u$

$$= \frac{1}{2}\tan 2x + C$$ $u = 2x$

6)

Solution 1: Substitute $u = z^2 + 1$.

$$\int \frac{2z\,dz}{\sqrt[3]{z^2+1}} = \int \frac{du}{u^{1/3}}$$ Let $u = z^2 + 1$, $du = 2z\,dz$.

$$= \int u^{-1/3}\,du$$ In the form $\int u^n\,du$

$$= \frac{u^{2/3}}{2/3} + C$$ Integrate with respect to u.

$$= \frac{3}{2}u^{2/3} + C$$

$$= \frac{3}{2}(z^2+1)^{2/3} + C$$ Replace u by $z^2 + 1$.

Solution 2: Substitute $u = \sqrt[3]{z^2+1}$ instead.

$$\int \frac{2z\,dz}{\sqrt[3]{z^2+1}} = \int \frac{3u^2\,du}{u} \qquad \text{Let } u = \sqrt[3]{z^2+1},\; u^3 = z^2+1,\; 3u^2\,du = 2z\,dz.$$

$$= 3\int u\,du$$

$$= 3\cdot\frac{u^2}{2} + C \qquad \text{Integrate with respect to } u.$$

$$= \frac{3}{2}(z^2+1)^{2/3} + C \qquad \text{Replace } u \text{ by } (z^2+1)^{1/3}.$$

7)

1. $\int \sin 3x\,dx, \quad u = 3x$

2. $\int x\sin(2x^2)\,dx, \quad u = 2x^2$

3. $\int \sec 2t \tan 2t\,dt, \quad u = 2t$ ##

4. $\int \left(1-\cos\frac{t}{2}\right)^2 \sin\frac{t}{2}\,dt, \quad u = 1-\cos\frac{t}{2}$ ##

5. $\int 28(7x-2)^{-5}\,dx, \quad u = 7x-2$

6. $\int x^3(x^4-1)^2\,dx, \quad u = x^4-1$

7. $\int \frac{9r^2\,dr}{\sqrt{1-r^3}}, \quad u = 1-r^3$

8. $\int 12(y^4+4y^2+1)^2(y^3+2y)\,dy, \quad u = y^4+4y^2+1$

9. $\int \sqrt{x}\sin^2(x^{3/2}-1)\,dx, \quad u = x^{3/2}-1$

10. $\int \frac{1}{x^2}\cos^2\left(\frac{1}{x}\right)dx, \quad u = -\frac{1}{x}$

11. $\int \csc^2 2\theta \cot 2\theta\,d\theta$

   a. Using $u = \cot 2\theta$ b. Using $u = \csc 2\theta$

12. $\int \frac{dx}{\sqrt{5x+8}}$

   a. Using $u = 5x+8$ b. Using $u = \sqrt{5x+8}$

13. $\int \sqrt{3-2s}\,ds$

14. $\int (2x+1)^3\,dx$

15. $\int \frac{1}{\sqrt{5s+4}}\,ds$

16. $\int \frac{3\,dx}{(2-x)^2}$

17. $\int \theta\sqrt[4]{1-\theta^2}\,d\theta$

18. $\int 8\theta\sqrt[3]{\theta^2-1}\,d\theta$

19. $\int 3y\sqrt{7-3y^2}\,dy$

20. $\int \frac{4y\,dy}{\sqrt{2y^2+1}}$

21. $\int \frac{1}{\sqrt{x}(1+\sqrt{x})^2}\,dx$

22. $\int \frac{(1+\sqrt{x})^3}{\sqrt{x}}\,dx$

23. $\int \cos(3z+4)\,dz$

24. $\int \sin(8z-5)\,dz$

25. $\int \sec^2(3x+2)\,dx$

26. $\int \tan^2 x \sec^2 x\,dx$

27. $\int \sin^5\frac{x}{3}\cos\frac{x}{3}\,dx$

28. $\int \tan^7\frac{x}{2}\sec^2\frac{x}{2}\,dx$

29. $\int r^2\left(\frac{r^3}{18}-1\right)^5 dr$

30. $\int r^4\left(7-\frac{r^5}{10}\right)^3 dr$

31. $\int x^{1/2}\sin(x^{3/2}+1)\,dx$

32. $\int x^{1/3}\sin(x^{4/3}-8)\,dx$

33. $\int \sec\left(v+\frac{\pi}{2}\right)\tan\left(v+\frac{\pi}{2}\right)dv$

34. $\int \csc\left(\frac{v-\pi}{2}\right)\cot\left(\frac{v-\pi}{2}\right)dv$

35. $\int \frac{\sin(2t+1)}{\cos^2(2t+1)}\,dt$

36. $\int \frac{6\cos t}{(2+\sin t)^3}\,dt$

37. $\int \sqrt{\cot y}\csc^2 y\, dy$

38. $\int \frac{\sec z \tan z}{\sqrt{\sec z}}\, dz$

39. $\int \frac{1}{t^2}\cos\left(\frac{1}{t} - 1\right) dt$

40. $\int \frac{1}{\sqrt{t}}\cos(\sqrt{t} + 3)\, dt$

41. $\int \frac{1}{\theta^2}\sin\frac{1}{\theta}\cos\frac{1}{\theta}\, d\theta$

42. $\int \frac{\cos\sqrt{\theta}}{\sqrt{\theta}\sin^2\sqrt{\theta}}\, d\theta$

43. $\int (s^3 + 2s^2 - 5s + 5)(3s^2 + 4s - 5)\, ds$

44. $\int (\theta^4 - 2\theta^2 + 8\theta - 2)(\theta^3 - \theta + 2)\, d\theta$

45. $\int t^3(1 + t^4)^3\, dt$

46. $\int \sqrt{\frac{x - 1}{x^5}}\, dx$

47. $\int x^3\sqrt{x^2 + 1}\, dx$

48. $\int 3x^5\sqrt{x^3 + 1}\, dx$

SOLUTIONS :

1. Let $u = 3x \Rightarrow du = 3\,dx \Rightarrow \frac{1}{3}\,du = dx$

$\int \sin 3x\, dx = \int \frac{1}{3}\sin u\, du = -\frac{1}{3}\cos u + C = -\frac{1}{3}\cos 3x + C$

2. Let $u = 2x^2 \Rightarrow du = 4x\,dx \Rightarrow \frac{1}{4}\,du = x\,dx$

$\int x\sin(2x^2)\, dx = \int \frac{1}{4}\sin u\, du = -\frac{1}{4}\cos u + C = -\frac{1}{4}\cos 2x^2 + C$

3. Let $u = 2t \Rightarrow du = 2\,dt \Rightarrow \frac{1}{2}\,du = dt$

$\int \sec 2t\tan 2t\, dt = \int \frac{1}{2}\sec u\tan u\, du = \frac{1}{2}\sec u + C = \frac{1}{2}\sec 2t + C$

4. Let $u = 1 - \cos\frac{t}{2} \Rightarrow du = \frac{1}{2}\sin\frac{t}{2}\,dt \Rightarrow 2\,du = \sin\frac{t}{2}\,dt$

$\int \left(1 - \cos\frac{t}{2}\right)^2\left(\sin\frac{t}{2}\right) dt = \int 2u^2\, du = \frac{2}{3}u^3 + C = \frac{2}{3}\left(1 - \cos\frac{t}{2}\right)^3 + C$

5. Let $u = 7x - 2 \Rightarrow du = 7\,dx \Rightarrow \frac{1}{7}\,du = dx$

$\int 28(7x-2)^{-5}\, dx = \int \frac{1}{7}(28)u^{-5}\, du = \int 4u^{-5}\, du = -u^{-4} + C = -(7x-2)^{-4} + C$

6. Let $u = x^4 - 1 \Rightarrow du = 4x^3\,dx \Rightarrow \frac{1}{4}\,du = x^3\,dx$

$\int x^3(x^4 - 1)^2\, dx = \int \frac{1}{4}u^2\, du = \frac{u^3}{12} + C = \frac{1}{12}(x^4 - 1)^3 + C$

7. Let $u = 1 - r^3 \Rightarrow du = -3r^2\,dr \Rightarrow -3\,du = 9r^2\,dr$
$\int \frac{9r^2\,dr}{\sqrt{1-r^3}} = \int -3u^{-1/2}\,du = -3(2)u^{1/2} + C = -6\left(1-r^3\right)^{1/2} + C$

8. Let $u = y^4 + 4y^2 + 1 \Rightarrow du = (4y^3 + 8y)\,dy \Rightarrow 3\,du = 12\left(y^3 + 2y\right)dy$
$\int 12\left(y^4 + 4y^2 + 1\right)^2\left(y^3 + 2y\right)dy = \int 3u^2\,du = u^3 + C = \left(y^4 + 4y^2 + 1\right)^3 + C$

9. Let $u = x^{3/2} - 1 \Rightarrow du = \frac{3}{2}x^{1/2}\,dx \Rightarrow \frac{2}{3}\,du = \sqrt{x}\,dx$
$\int \sqrt{x}\sin^2\left(x^{3/2} - 1\right)dx = \int \frac{2}{3}\sin^2 u\,du = \frac{2}{3}\left(\frac{u}{2} - \frac{1}{4}\sin 2u\right) + C = \frac{1}{3}\left(x^{3/2} - 1\right) - \frac{1}{6}\sin\left(2x^{3/2} - 2\right) + C$

10. Let $u = -\frac{1}{x} \Rightarrow du = \frac{1}{x^2}\,dx$
$\int \frac{1}{x^2}\cos^2\left(\frac{1}{x}\right)dx = \int \cos^2(-u)\,du = \int \cos^2(u)\,du = \left(\frac{u}{2} + \frac{1}{4}\sin 2u\right) + C = -\frac{1}{2x} + \frac{1}{4}\sin\left(-\frac{2}{x}\right) + C$
$= -\frac{1}{2x} - \frac{1}{4}\sin\left(\frac{2}{x}\right) + C$

11. (a) Let $u = \cot 2\theta \Rightarrow du = -2\csc^2 2\theta\,d\theta \Rightarrow -\frac{1}{2}\,du = \csc^2 2\theta\,d\theta$
$\int \csc^2 2\theta \cot 2\theta\,d\theta = -\int \frac{1}{2}u\,du = -\frac{1}{2}\left(\frac{u^2}{2}\right) + C = -\frac{u^2}{4} + C = -\frac{1}{4}\cot^2 2\theta + C$
(b) Let $u = \csc 2\theta \Rightarrow du = -2\csc 2\theta\cot 2\theta\,d\theta \Rightarrow -\frac{1}{2}\,du = \csc 2\theta\cot 2\theta\,d\theta$
$\int \csc^2 2\theta\cot 2\theta\,d\theta = \int -\frac{1}{2}u\,du = -\frac{1}{2}\left(\frac{u^2}{2}\right) + C = -\frac{u^2}{4} + C = -\frac{1}{4}\csc^2 2\theta + C$

12. (a) Let $u = 5x + 8 \Rightarrow du = 5\,dx \Rightarrow \frac{1}{5}\,du = dx$
$\int \frac{dx}{\sqrt{5x+8}} = \int \frac{1}{5}\left(\frac{1}{\sqrt{u}}\right)du = \frac{1}{5}\int u^{-1/2}\,du = \frac{1}{5}\left(2u^{1/2}\right) + C = \frac{2}{5}u^{1/2} + C = \frac{2}{5}\sqrt{5x+8} + C$
(b) Let $u = \sqrt{5x+8} \Rightarrow du = \frac{1}{2}(5x+8)^{-1/2}(5)\,dx \Rightarrow \frac{2}{5}\,du = \frac{dx}{\sqrt{5x+8}}$
$\int \frac{dx}{\sqrt{5x+8}} = \int \frac{2}{5}\,du = \frac{2}{5}u + C = \frac{2}{5}\sqrt{5x+8} + C$

13. Let $u = 3 - 2s \Rightarrow du = -2\,ds \Rightarrow -\frac{1}{2}\,du = ds$
$\int \sqrt{3-2s}\,ds = \int \sqrt{u}\left(-\frac{1}{2}\,du\right) = -\frac{1}{2}\int u^{1/2}\,du = \left(-\frac{1}{2}\right)\left(\frac{2}{3}u^{3/2}\right) + C = -\frac{1}{3}(3-2s)^{3/2} + C$

14. Let $u = 2x + 1 \Rightarrow du = 2\,dx \Rightarrow \frac{1}{2}\,du = dx$
$\int (2x+1)^3\,dx = \int u^3\left(\frac{1}{2}\,du\right) = \frac{1}{2}\int u^3\,du = \left(\frac{1}{2}\right)\left(\frac{u^4}{4}\right) + C = \frac{1}{8}(2x+1)^4 + C$

15. Let $u = 5s + 4 \Rightarrow du = 5\,ds \Rightarrow \frac{1}{5}\,du = ds$
$\int \frac{1}{\sqrt{5s+4}}\,ds = \int \frac{1}{\sqrt{u}}\left(\frac{1}{5}\,du\right) = \frac{1}{5}\int u^{-1/2}\,du = \left(\frac{1}{5}\right)\left(2u^{1/2}\right) + C = \frac{2}{5}\sqrt{5s+4} + C$

16. Let $u = 2 - x \Rightarrow du = -dx \Rightarrow -du = dx$
$\int \frac{3}{(2-x)^2}\,dx = \int \frac{3(-du)}{u^2} = -3\int u^{-2}\,du = -3\left(\frac{u^{-1}}{-1}\right) + C = \frac{3}{2-x} + C$

17. Let $u = 1 - \theta^2 \Rightarrow du = -2\theta\,d\theta \Rightarrow -\frac{1}{2}\,du = \theta\,d\theta$
$\int \theta\sqrt[4]{1-\theta^2}\,d\theta = \int \sqrt[4]{u}\left(-\frac{1}{2}\,du\right) = -\frac{1}{2}\int u^{1/4}\,du = \left(-\frac{1}{2}\right)\left(\frac{4}{5}u^{5/4}\right) + C = -\frac{2}{5}\left(1-\theta^2\right)^{5/4} + C$

18. Let $u = \theta^2 - 1 \Rightarrow du = 2\theta\,d\theta \Rightarrow 4\,du = 8\theta\,d\theta$
$\int 8\theta\sqrt[3]{\theta^2 - 1}\,d\theta = \int \sqrt[3]{u}\,(4\,du) = 4\int u^{1/3}\,du = 4\left(\frac{3}{4}u^{4/3}\right) + C = 3\left(\theta^2 - 1\right)^{4/3} + C$

19. Let $u = 7 - 3y^2 \Rightarrow du = -6y\,dy \Rightarrow -\frac{1}{2}\,du = 3y\,dy$

$$\int 3y\sqrt{7-3y^2}\,dy = \int \sqrt{u}\left(-\frac{1}{2}\,du\right) = -\frac{1}{2}\int u^{1/2}\,du = \left(-\frac{1}{2}\right)\left(\frac{2}{3}u^{3/2}\right) + C = -\frac{1}{3}\left(7-3y^2\right)^{3/2} + C$$

20. Let $u = 2y^2 + 1 \Rightarrow du = 4y\,dy$

$$\int \frac{4y\,dy}{\sqrt{2y^2+1}} = \int \frac{1}{\sqrt{u}}\,du = \int u^{-1/2}\,du = 2u^{1/2} + C = 2\sqrt{2y^2+1} + C$$

21. Let $u = 1 + \sqrt{x} \Rightarrow du = \frac{1}{2\sqrt{x}}\,dx \Rightarrow 2\,du = \frac{1}{\sqrt{x}}\,dx$

$$\int \frac{1}{\sqrt{x}\left(1+\sqrt{x}\right)^2}\,dx = \int \frac{2\,du}{u^2} = -\frac{2}{u} + C = \frac{-2}{1+\sqrt{x}} + C$$

22. Let $u = 1 + \sqrt{x} \Rightarrow du = \frac{1}{2\sqrt{x}}\,dx \Rightarrow 2\,du = \frac{1}{\sqrt{x}}\,dx$

$$\int \frac{\left(1+\sqrt{x}\right)^3}{\sqrt{x}}\,dx = \int u^3\,(2\,du) = 2\left(\frac{1}{4}u^4\right) + C = \frac{1}{2}\left(1+\sqrt{x}\right)^4 + C$$

23. Let $u = 3z + 4 \Rightarrow du = 3\,dz \Rightarrow \frac{1}{3}\,du = dz$

$$\int \cos(3z+4)\,dz = \int (\cos u)\left(\frac{1}{3}\,du\right) = \frac{1}{3}\int \cos u\,du = \frac{1}{3}\sin u + C = \frac{1}{3}\sin(3z+4) + C$$

24. Let $u = 8z - 5 \Rightarrow du = 8\,dz \Rightarrow \frac{1}{8}\,du = dz$

$$\int \sin(8z-5)\,dz = \int (\sin u)\left(\frac{1}{8}\,du\right) = \frac{1}{8}\int \sin u\,du = \frac{1}{8}(-\cos u) + C = -\frac{1}{8}\cos(8z-5) + C$$

25. Let $u = 3x + 2 \Rightarrow du = 3\,dx \Rightarrow \frac{1}{3}\,du = dx$

$$\int \sec^2(3x+2)\,dx = \int (\sec^2 u)\left(\frac{1}{3}\,du\right) = \frac{1}{3}\int \sec^2 u\,du = \frac{1}{3}\tan u + C = \frac{1}{3}\tan(3x+2) + C$$

26. Let $u = \tan x \Rightarrow du = \sec^2 x\,dx$

$$\int \tan^2 x\sec^2 x\,dx = \int u^2\,du = \frac{1}{3}u^3 + C = \frac{1}{3}\tan^3 x + C$$

27. Let $u = \sin\left(\frac{x}{3}\right) \Rightarrow du = \frac{1}{3}\cos\left(\frac{x}{3}\right)dx \Rightarrow 3\,du = \cos\left(\frac{x}{3}\right)dx$

$$\int \sin^5\left(\frac{x}{3}\right)\cos\left(\frac{x}{3}\right)dx = \int u^5\,(3\,du) = 3\left(\frac{1}{6}u^6\right) + C = \frac{1}{2}\sin^6\left(\frac{x}{3}\right) + C$$

28. Let $u = \tan\left(\frac{x}{2}\right) \Rightarrow du = \frac{1}{2}\sec^2\left(\frac{x}{2}\right)dx \Rightarrow 2\,du = \sec^2\left(\frac{x}{2}\right)dx$

$$\int \tan^7\left(\frac{x}{2}\right)\sec^2\left(\frac{x}{2}\right)dx = \int u^7\,(2\,du) = 2\left(\frac{1}{8}u^8\right) + C = \frac{1}{4}\tan^8\left(\frac{x}{2}\right) + C$$

29. Let $u = \frac{r^3}{18} - 1 \Rightarrow du = \frac{r^2}{6}\,dr \Rightarrow 6\,du = r^2\,dr$

$$\int r^2\left(\frac{r^3}{18}-1\right)^5 dr = \int u^5\,(6\,du) = 6\int u^5\,du = 6\left(\frac{u^6}{6}\right) + C = \left(\frac{r^3}{18}-1\right)^6 + C$$

30. Let $u = 7 - \frac{r^5}{10} \Rightarrow du = -\frac{1}{2}r^4\,dr \Rightarrow -2\,du = r^4\,dr$

$$\int r^4\left(7-\frac{r^5}{10}\right)^3 dr = \int u^3\,(-2\,du) = -2\int u^3\,du = -2\left(\frac{u^4}{4}\right) + C = -\frac{1}{2}\left(7-\frac{r^5}{10}\right)^4 + C$$

31. Let $u = x^{3/2} + 1 \Rightarrow du = \frac{3}{2}x^{1/2}\,dx \Rightarrow \frac{2}{3}\,du = x^{1/2}\,dx$

$$\int x^{1/2}\sin\left(x^{3/2}+1\right)dx = \int (\sin u)\left(\frac{2}{3}\,du\right) = \frac{2}{3}\int \sin u\,du = \frac{2}{3}(-\cos u) + C = -\frac{2}{3}\cos\left(x^{3/2}+1\right) + C$$

32. Let $u = x^{4/3} - 8 \Rightarrow du = \frac{4}{3} x^{1/3}\, dx \Rightarrow \frac{3}{4}\, du = x^{1/3}\, dx$
$\int x^{1/3} \sin\left(x^{4/3} - 8\right) dx = \int (\sin u)\left(\frac{3}{4}\, du\right) = \frac{3}{4}\int \sin u\, du = \frac{3}{4}(-\cos u) + C = -\frac{3}{4}\cos\left(x^{4/3} - 8\right) + C$

33. Let $u = \sec\left(v + \frac{\pi}{2}\right) \Rightarrow du = \sec\left(v + \frac{\pi}{2}\right)\tan\left(v + \frac{\pi}{2}\right) dv$
$\int \sec\left(v + \frac{\pi}{2}\right)\tan\left(v + \frac{\pi}{2}\right) dv = \int du = u + C = \sec\left(v + \frac{\pi}{2}\right) + C$

34. Let $u = \csc\left(\frac{v-\pi}{2}\right) \Rightarrow du = -\frac{1}{2}\csc\left(\frac{v-\pi}{2}\right)\cot\left(\frac{v-\pi}{2}\right) dv \Rightarrow -2\, du = \csc\left(\frac{v-\pi}{2}\right)\cot\left(\frac{v-\pi}{2}\right) dv$
$\int \csc\left(\frac{v-\pi}{2}\right)\cot\left(\frac{v-\pi}{2}\right) dv = \int -2\, du = -2u + C = -2\csc\left(\frac{v-\pi}{2}\right) + C$

35. Let $u = \cos(2t+1) \Rightarrow du = -2\sin(2t+1)\, dt \Rightarrow -\frac{1}{2}\, du = \sin(2t+1)\, dt$
$\int \frac{\sin(2t+1)}{\cos^2(2t+1)}\, dt = \int -\frac{1}{2}\frac{du}{u^2} = \frac{1}{2u} + C = \frac{1}{2\cos(2t+1)} + C$

36. Let $u = 2 + \sin t \Rightarrow du = \cos t\, dt$
$\int \frac{6\cos t}{(2+\sin t)^3}\, dt = \int \frac{6}{u^3}\, du = 6\int u^{-3}\, du = 6\left(\frac{u^{-2}}{-2}\right) + C = -3(2+\sin t)^{-2} + C$

37. Let $u = \cot y \Rightarrow du = -\csc^2 y\, dy \Rightarrow -du = \csc^2 y\, dy$
$\int \sqrt{\cot y}\,\csc^2 y\, dy = \int \sqrt{u}\,(-du) = -\int u^{1/2}\, du = -\frac{2}{3}u^{3/2} + C = -\frac{2}{3}(\cot y)^{3/2} + C = -\frac{2}{3}\left(\cot^3 y\right)^{1/2} + C$

38. Let $u = \sec z \Rightarrow du = \sec z \tan z\, dz$
$\int \frac{\sec z \tan z}{\sqrt{\sec z}}\, dz = \int \frac{1}{\sqrt{u}}\, du = \int u^{-1/2}\, du = 2u^{1/2} + C = 2\sqrt{\sec z} + C$

39. Let $u = \frac{1}{t} - 1 = t^{-1} - 1 \Rightarrow du = -t^{-2}\, dt \Rightarrow -du = \frac{1}{t^2}\, dt$
$\int \frac{1}{t^2}\cos\left(\frac{1}{t} - 1\right) dt = \int (\cos u)(-du) = -\int \cos u\, du = -\sin u + C = -\sin\left(\frac{1}{t} - 1\right) + C$

40. Let $u = \sqrt{t} + 3 = t^{1/2} + 3 \Rightarrow du = \frac{1}{2}t^{-1/2}\, dt \Rightarrow 2\, du = \frac{1}{\sqrt{t}}\, dt$
$\int \frac{1}{\sqrt{t}}\cos\left(\sqrt{t} + 3\right) dt = \int (\cos u)(2\, du) = 2\int \cos u\, du = 2\sin u + C = 2\sin\left(\sqrt{t} + 3\right) + C$

41. Let $u = \sin\frac{1}{\theta} \Rightarrow du = \left(\cos\frac{1}{\theta}\right)\left(-\frac{1}{\theta^2}\right) d\theta \Rightarrow -du = \frac{1}{\theta^2}\cos\frac{1}{\theta}\, d\theta$
$\int \frac{1}{\theta^2}\sin\frac{1}{\theta}\cos\frac{1}{\theta}\, d\theta = \int -u\, du = -\frac{1}{2}u^2 + C = -\frac{1}{2}\sin^2\frac{1}{\theta} + C$

42. Let $u = \csc\sqrt{\theta} \Rightarrow du = \left(-\csc\sqrt{\theta}\cot\sqrt{\theta}\right)\left(\frac{1}{2\sqrt{\theta}}\right) d\theta \Rightarrow -2\, du = \frac{1}{\sqrt{\theta}}\cot\sqrt{\theta}\csc\sqrt{\theta}\, d\theta$
$\int \frac{\cos\sqrt{\theta}}{\sqrt{\theta}\sin^2\sqrt{\theta}}\, d\theta = \int \frac{1}{\sqrt{\theta}}\cot\sqrt{\theta}\csc\sqrt{\theta}\, d\theta = \int -2\, du = -2u + C = -2\csc\sqrt{\theta} + C = -\frac{2}{\sin\sqrt{\theta}} + C$

43. Let $u = s^3 + 2s^2 - 5s + 5 \Rightarrow du = (3s^2 + 4s - 5)\, ds$
$\int (s^3 + 2s^2 - 5s + 5)(3s^2 + 4s - 5)\, ds = \int u\, du = \frac{u^2}{2} + C = \frac{(s^3 + 2s^2 - 5s + 5)^2}{2} + C$

44. Let $u = \theta^4 - 2\theta^2 + 8\theta - 2 \Rightarrow du = (4\theta^3 - 4\theta + 8)\, d\theta \Rightarrow \frac{1}{4}\, du = (\theta^3 - \theta + 2)\, d\theta$
$\int (\theta^4 - 2\theta^2 + 8\theta - 2)(\theta^3 - \theta + 2)\, d\theta = \int u\left(\frac{1}{4}\, du\right) = \frac{1}{4}\int u\, du = \frac{1}{4}\left(\frac{u^2}{2}\right) + C = \frac{(\theta^4 - 2\theta^2 + 8\theta - 2)^2}{8} + C$

45. Let $u = 1 + t^4 \Rightarrow du = 4t^3\,dt \Rightarrow \frac{1}{4}\,du = t^3\,dt$

$\int t^3\,(1+t^4)^3\,dt = \int u^3\left(\frac{1}{4}\,du\right) = \frac{1}{4}\left(\frac{1}{4}u^4\right) + C = \frac{1}{16}(1+t^4)^4 + C$

46. Let $u = 1 - \frac{1}{x} \Rightarrow du = \frac{1}{x^2}\,dx$

$\int \sqrt{\frac{x-1}{x^5}}\,dx = \int \frac{1}{x^2}\sqrt{\frac{x-1}{x}}\,dx = \int \frac{1}{x^2}\sqrt{1-\frac{1}{x}}\,dx = \int \sqrt{u}\,du = \int u^{1/2}\,du = \frac{2}{3}u^{3/2} + C = \frac{2}{3}\left(1-\frac{1}{x}\right)^{3/2} + C$

47. Let $u = x^2 + 1$. Then $du = 2x dx$ and $\frac{1}{2}du = x dx$ and $x^2 = u - 1$. Thus $\int x^3\sqrt{x^2+1}\,dx = \int (u-1)\frac{1}{2}\sqrt{u}\,du$

$= \frac{1}{2}\int\left(u^{3/2} - u^{1/2}\right)du = \frac{1}{2}\left[\frac{2}{5}u^{5/2} - \frac{2}{3}u^{3/2}\right] + C = \frac{1}{5}u^{5/2} - \frac{1}{3}u^{3/2} + C = \frac{1}{5}(x^2+1)^{5/2} - \frac{1}{3}(x^2+1)^{3/2} + C$

48. Let $u = x^3 + 1 \Rightarrow du = 3x^2 dx$ and $x^3 = u - 1$. So $\int 3x^5\sqrt{x^3+1}\,dx = \int (u-1)\sqrt{u}\,du = \int\left(u^{3/2} - u^{1/2}\right)du$

$= \frac{2}{5}u^{5/2} - \frac{2}{3}u^{3/2} + C = \frac{2}{5}(x^3+1)^{5/2} - \frac{2}{3}(x^3+1)^{3/2} + C$