Definition 12 Standard deviation If $X$ is a random variable, the standard deviation of $X$, denoted by $\sigma_X$, is defined as $+\sqrt{\text{var} \ [X]}$. \\

The standard deviation of a random variable, like the variance, is a measure of the spread or dispersion of the values of the random variable. In many applications it is preferable to the variance as such a measure since it will have the same measurement units as the random variable itself.

EXAMPLE 14 Let $X$ be the total of the two dice in the experiment of tossing two dice.

$$\text{var} \ [X] = \sum (x_j - \mu_X)^2 f_X(x_j)$$

$$= (2 - 7)^2 \frac{1}{36} + (3 - 7)^2 \frac{2}{36} + (4 - 7)^2 \frac{3}{36} + (5 - 7)^2 \frac{4}{36}$$

$$+ (6 - 7)^2 \frac{5}{36} + (7 - 7)^2 \frac{6}{36} + (8 - 7)^2 \frac{7}{36} + (9 - 7)^2 \frac{8}{36}$$

$$+ (10 - 7)^2 \frac{9}{36} + (11 - 7)^2 \frac{10}{36} + (12 - 7)^2 \frac{11}{36} = \frac{2}{36}.$$ \\

EXAMPLE 15 Let $X$ be a random variable with probability density given by $f_X(x) = \lambda e^{-\lambda x} I_{(0, \infty)}(x)$; then

$$\text{Var} \ [X] = \int_{-\infty}^{\infty} (x - \mu_X)^2 f_X(x) \, dx$$

$$= \int_{0}^{\infty} \left( x - \frac{1}{\lambda} \right)^2 \lambda e^{-\lambda x} \, dx$$

$$= \frac{1}{\lambda^2}.$$ \\

EXAMPLE 16 Let $X$ be a random variable with cumulative distribution given by $F_X(x) = (1 - pe^{-\lambda x}) I_{(0, \infty)}(x)$; then

$$\text{Var} \ [X] = \int_{0}^{\infty} 2x[1 - F(x) + F(-x)] \, dx - \mu_X^2$$

$$= \int_{0}^{\infty} 2xpe^{-\lambda x} \, dx - \left( \frac{p}{\lambda} \right)^2$$

$$= 2 \frac{p}{\lambda^2} - \left( \frac{p}{\lambda} \right)^2 = \frac{p(2 - p)}{\lambda^2}.$$
4.3 Expected Value of a Function of a Random Variable

We defined the expectation of an arbitrary random variable $X$, called the mean of $X$, in Subsec. 4.1. In this subsection, we will define the expectation of a function of a random variable for discrete or continuous random variables.

**Definition 13** Expectation Let $X$ be a random variable and $g(\cdot)$ be a function with both domain and counterdomain the real line. The expectation or expected value of the function $g(\cdot)$ of the random variable $X$, denoted by $\mathbb{E}[g(X)]$, is defined by:

(i) \[ \mathbb{E}[g(X)] = \sum_j g(x_j)f_X(x_j) \] (10)

if $X$ is discrete with mass points $x_1, x_2, \ldots, x_j, \ldots$ (provided this series is absolutely convergent).

(ii) \[ \mathbb{E}[g(X)] = \int_{-\infty}^{\infty} g(x)f_X(x) \, dx \] (11)

if $X$ is continuous with probability density function $f_X(x)$ (provided $\int_{-\infty}^{\infty} |g(x)|f_X(x) \, dx < \infty$)*

**Remark** If $g(x) = x$, then $\mathbb{E}[g(X)] = \mathbb{E}[X]$ is the mean of $X$. If $g(x) = (x - \mu_X)^2$, then $\mathbb{E}[g(X)] = \mathbb{E}[(X - \mu_X)^2] = \text{var}[X]$.

**Theorem 3** Below are properties of expected value:

(i) $\mathbb{E}[c] = c$ for a constant $c$.

(ii) $\mathbb{E}[cg(X)] = c\mathbb{E}[g(X)]$ for a constant $c$.

(iii) $\mathbb{E}[c_1g_1(X) + c_2g_2(X)] = c_1\mathbb{E}[g_1(X)] + c_2\mathbb{E}[g_2(X)]$

(iv) $\mathbb{E}[g_1(X)] \leq \mathbb{E}[g_2(X)]$ if $g_1(x) \leq g_2(x)$ for all $x$.

**Theorem 4** If $X$ is a random variable, $\text{var}[X] = \mathbb{E}[(X - \mathbb{E}[X])^2] = \mathbb{E}[X^2] - (\mathbb{E}[X])^2$ provided $\mathbb{E}[X^2]$ exists.
Theorem 5  Let $X$ be a random variable and $g(\cdot)$ a nonnegative function with domain the real line; then

$$P[g(X) \geq k] \leq \frac{\sigma[g(X)]}{k} \quad \text{for every } k > 0. \quad (12)$$

**Proof**  Assume that $X$ is a continuous random variable with probability density function $f_X(\cdot)$; then

$$\sigma[g(X)] = \int_{-\infty}^{\infty} g(x)f_X(x) \, dx = \int_{\{x: g(x) \geq k\}} g(x)f_X(x) \, dx$$

$$+ \int_{\{x: g(x) < k\}} g(x)f_X(x) \, dx \geq \int_{\{x: g(x) \geq k\}} g(x)f_X(x) \, dx$$

$$\geq \int_{\{x: g(x) \geq k\}} kf_X(x) \, dx = kP[g(X) \geq k].$$

Divide by $k$, and the result follows. A similar proof holds for $X$ discrete.

Corollary  Chebyshev inequality  If $X$ is a random variable with finite variance,

$$P[|X - \mu_X| \geq r\sigma_X] = P[(X - \mu_X)^2 \geq r^2\sigma_X^2] \leq \frac{1}{r^2} \quad \text{for every } r > 0. \quad (13)$$

**Proof**  Take $g(x) = (x - \mu_X)^2$ and $k = r^2\sigma_X^2$ in Eq. (12) of Theorem 5.