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1.1 Basic probability

Probability or **chance** can be measured on a scale which runs from **zero**, which represents **impossibility**, to **one**, which represents **certainty**.

1.1.1 Terminology

A **sample space**, Ω, is the set of all possible **outcomes** of an experiment. An **event** $E \in \Omega$ is a subset of Ω.

Example 1 *Experiment:* roll a die twice. Possible *events* are $E_1 = \{1\text{st face is a } 6\}$, $E_2 = \{\text{sum of faces} = 3\}$, $E_3 = \{\text{sum of faces is odd}\}$, $E_4 = \{1\text{st face - 2nd face} = 3\}$. Identify the sample space and the above events. Obtain their probabilities when the die is **fair**.

Answer:

		second roll					
	1	(1,1)	(1,2)	(1,3)	(1,4)	(1,5)	(1,6)
	2	(2,1)	(2,2)	(2,3)	(2,4)	(2,5)	(2,6)
first	3	(3,1)	(3,2)	(3,3)	(3,4)	(3,5)	(3,6)
roll	4	(4,1)	(4,2)	(4,3)	(4,4)	(4,5)	(4,6)
	5	(5,1)	(5,2)	(5,3)	(5,4)	(5,5)	(5,6)
	6	(6,1)	(6,2)	(6,3)	(6,4)	(6,5)	(6,6)

$p(E_1) = \frac{1}{6}; p(E_2) = \frac{1}{18}; p(E_3) = \frac{1}{2}; p(E_4) = \frac{1}{12}.$

Combinations of events

Given events A and B, further events can be identified as follows.

- The **complement** of any event A, written $\bar{A}$ or A^c, means that A does **not** occur.
- The **union** of any two events A and B, written $A \cup B$, means that A **or** B **or** both occur.
- The **intersection** of A and B, written as $A \cap B$, means that both A **and** B **occur**

1.1.2 Probability axioms

Let $\mathcal{F}$ be the class of all events in Ω. A **probability (measure)** P on $(\Omega, \mathcal{F})$ is a real-valued function satisfying the following three axioms:

1. $P(E) \geq 0$ for every $E \in \mathcal{F}$
2. $P(\Omega) = 1$
3. Suppose the events E_1 and E_2 are mutually exclusive (that is, $E_1 \cap E_2 = \emptyset$).

Then

$$P(E_1 \cup E_2) = P(E_1) + P(E_2)$$

Some consequences:

(i) $P(\bar{E}) = 1 - P(E)$ (so in particular $P(\emptyset) = 0$)

(ii) For *any* two events E_1 and E_2 we have the **addition rule**

$$P(E_1 \cup E_2) = P(E_1) + P(E_2) - P(E_1 \cap E_2)$$

Example 1: (continued)

Obtain $P(E_1 \cap E_2)$, $P(E_1 \cup E_2)$, $P(E_1 \cap E_3)$ and $P(E_1 \cup E_3)$

Answer: $P(E_1 \cap E_2) = P(\emptyset) = 0$

$P(E_1 \cup E_2) = P(E_1) + P(E_2) = \frac{1}{6} + \frac{1}{18} = \frac{2}{9}$

$P(E_1 \cap E_3) = P(6,1),(6,3),(6,5) = \frac{3}{36} = \frac{1}{12}$

$P(E_1 \cup E_3) = P(E_1) + P(E_3) - P(E_1 \cap E_3) = \frac{1}{6} + \frac{1}{2} - \frac{1}{12} = \frac{7}{12}$

[*Notes on axioms:*

(1) In order to cope with infinite sequences of events, it is necessary to strengthen axiom 3 to

3'. $P(\cup_{i=1}^{\infty}) = \sum_{i=1}^{\infty} P(E_i)$ for any sequence $(E_1, E_2, \cdots)$ of mutually exclusive events.

(2) When Ω is noncountably infinite, in order to make the theory rigorous it is usually necessary to restrict the class of events $\mathcal{F}$ to which probabilities are assigned.]

1.1.3 Conditional probability

Supose $P(E_2) \neq 0$. The **conditional probability** of the event E_1 given E_2 is defined as

$$P(E_1|E_2) = \frac{P(E_1 \cap E_2)}{P(E_2)}.$$

The conditional probability is undefined if $P(E_2) = 0$. The conditional probability formula above yields the **multiplication rule**:

$$\begin{aligned} P(E_1 \cap E_2) &= P(E_1)P(E_2|E_1) \\ &= P(E_2)P(E_1|E_2) \end{aligned}$$

Independence

Events E_1 and E_2 are said to be **independent** if

$$P(E_1 \cap E_2) = P(E_1)P(E_2).$$

Note that this implies that $P(E_1|E_2) = P(E_1)$ and $P(E_2|E_1) = P(E_2)$. Thus knowledge of the occurrence of one of the events does not affect the likelihood of occurrence of the other.

Events $E_1, \ldots, E_k$ are **pairwise independent** if $P(E_i \cap E_j) = P(E_i)P(E_j)$ for all $i \neq j$. They are **mutually independent** if for all subsets $P(\cap_j E_j) = \prod_j P(E_j)$. Clearly, mutual independence $\Rightarrow$ pairwise independence, but the converse is false (see question 4 of the self study exercises).

Example 1 (continued): Find $P(E_1|E_2)$ and $P(E_1|E_3)$. Are E_1, E_2 independent?

Answer: $P(E_1|E_2) = \frac{P(E_1 \cap E_2)}{P(E_2)} = 0$, $P(E_1|E_3) = \frac{P(E_1 \cap E_3)}{P(E_3)} = \frac{1/12}{1/2} = \frac{1}{6}$

$P(E_1)P(E_2) \neq 0$ so $P(E_1 \cap E_2) \neq P(E_1)P(E_2)$ and thus E_1 and E_2 are not independent.

Law of total probability (partition law)

Suppose that $B_1, \ldots, B_k$ are **mutually exclusive** and **exhaustive** events (*i.e.* $B_i \cap B_j = \emptyset$ for all $i \neq j$ and $\cup_i B_i = \Omega$).

Let A be any event. Then

$$P(A) = \sum_{j=1}^{k} P(A|B_j)P(B_j)$$

Bayes' Rule

Suppose that events $B_1, \ldots, B_k$ are mutually exclusive and exhaustive and let A be any event. Then

$$P(B_j|A) = \frac{P(A|B_j)P(B_j)}{P(A)} = \frac{P(A|B_j)P(B_j)}{\sum_i P(A|B_i)P(B_i)}$$

Example 2: (Cancer diagnosis) A screening programme for a certain type of cancer has reliabilities $P(A|D) = 0.98$, $P(A|\bar{D}) = 0.05$, where D is the event "disease is present" and A is the event "test gives a positive result". It is known that 1 in 10, 000 of the population has the disease. Suppose that an individual's test result is positive. What is the probability that that person has the disease?

Answer: We require $P(D|A)$. First find $P(A)$.

$P(A) = P(A|D)P(D) + P(A|\bar{D})P(\bar{D}) = 0.98 \times 0.0001 + 0.05 \times 0.9999 = 0.050093.$

By Bayes' rule; $P(D|A) = \frac{P(A|D)P(D)}{P(A)} = \frac{0.0001 \times 0.98}{0.050093} = 0.002.$

The person is still very unlikely to have the disease even though the test is positive.

Example 3: (Bertrand's Box Paradox) Three indistinguishable boxes contain black and white beads as shown: [ww], [wb], [bb]. A box is chosen at random and a bead chosen at random from the selected box. What is the probability of that the [wb] box was chosen given that selected bead was white?

Answer: $E \equiv$ 'chose the [wb] box', $W \equiv$ 'selected bead is white'. By the partition law: $P(W) = 1 \times \frac{1}{3} + \frac{1}{2} \times \frac{1}{3} + 0 \times \frac{1}{3} = \frac{1}{2}$. Now using Bayes' rule $P(E|W) = \frac{P(E)P(W|E)}{P(W)} = \frac{\frac{1}{3} \times \frac{1}{2}}{\frac{1}{2}} = \frac{1}{3}$ (i.e. even though a bead from the selected box has been seen, the probability that the box is [wb] is still $\frac{1}{3}$).