Lecturer three

General Concept in Q. M.

3-1 Probability Theorem

Because of the statistical interpretation, probability plays a central role in quantum mechanics, so I digress now for a brief discussion of the theory of probability. It is mainly a question of introducing some notation and terminology, and I shall do it in the context of a simple example.

Imagine a room containing 14 people, whose ages are as follows:

One person aged 14, *one* person aged 15, *three* people aged 16, *two* people aged 22, *two* people aged 24, and *five* people aged 25.

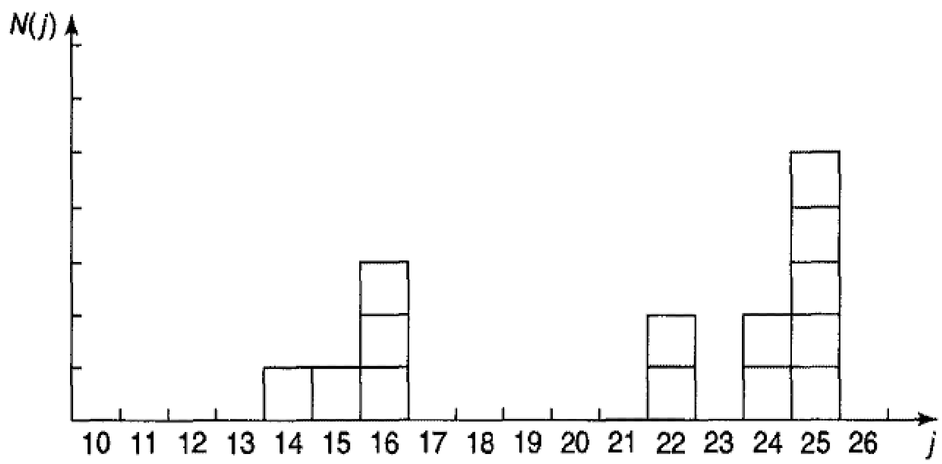

Figure 1.4: Histogram showing the number of people, $N(j)$, with age j, for the example in Section 1.3.

The total number of students, by the way, is given by:

$$N = \sum_{j=0}^{\infty} N(j) \qquad 3-1$$

Where N (j) = number of times the value j occurs.

To find the mean or average value (value of age):

$$\langle j\rangle = \frac{\sum j\,N(j)}{N} = \sum_{j=0}^{\infty} jP(j) \qquad 3-2$$

Where *p (j)* is the ***probability***

$$\langle j\rangle = \frac{(14)+(15)+3(16)+2(22)+2(24)+5(25)}{14} = \frac{294}{14} = 21$$

The average of the squares of the ages:

$$\langle j^2\rangle = \sum_{j=0}^{\infty} j^2\,p(j) \qquad 3-3$$

In general, the average value of some function of *j* is given by:

$$\langle f(j)\rangle = \sum_{j=0}^{\infty} f(j)\,p(j) \qquad 3-4$$

$$\sigma^2 = \langle(\Delta j)^2\rangle. \qquad 3-5$$

This quantity is known as the variance of the distribution; σ itself (the square root of the average of the square of the deviation from the average-gulp!) is called the standard deviation. The latter is the customary measure of the spread about $< j >$. There is a useful little theorem involving *standard deviations* σ^2:

$$\sigma^2 = \langle j^2\rangle - \langle j\rangle^2 \qquad 3-6$$

Where 3-5 provides faster methods for computing σ

Since σ^2 is plainly nonnegative $\langle j^2 \rangle \geq \langle j \rangle^2$ and the two are equal only when a = 0, which is to say, for distributions with no spread at all (every member having the same value).

Probability that individual (chosen at random) lies between x and (x+dx)

$$Probability = \rho(x)dx \qquad 3-7$$

The proportionality factor, ρ(x), is often loosely called "the probability of getting x," but this is sloppy language; a better term is *probability density*. The probability that x lies between *a* and *b* (a finite interval) is given by the integral of ρ(x):

$$P_{ab} = \int_a^b \rho(x)dx \qquad 3-8$$

And the rules we deduced for discrete distributions translate in the obvious way:

$$\int_{-\infty}^{\infty} \rho(x)dx = 1 \qquad 3-9$$

$$\langle x \rangle = \int_{-\infty}^{\infty} x\rho(x)dx \qquad 3-10$$

$$\langle f(x) \rangle = \int_{-\infty}^{\infty} f(x)\rho(x)dx \qquad 3-11$$

$$\sigma^2 \equiv \langle (\Delta x)^2 \rangle = \langle x^2 \rangle - \langle x \rangle^2 \qquad 3-12$$

3-2 Normalization

The probability of finding the particle between x and (x+dx), at time t:

$$|\Psi(x,t)|^2 dx \qquad 3-13$$

It follows (Equation 1.16) that the integral of $|\Psi|^2$ must be 1 (the particle's got to be somewhere):

$$\int_{-\infty}^{\infty} |\Psi(\mathrm{x},\mathrm{t})|^2 \, dx = 1 \qquad 3-14$$

The wave function in this case is called "*normalized wave function*", and above equation is called "*normalization condition*".

If the wave function is not normalized, then it must be multiplied by a multiplicative constant N, and using the normalization to find N.

3-3 Probability Current density

Schrödinger equation:

$$-\frac{\hbar^2}{2m}\frac{\partial^2\Psi(\mathrm{x,t})}{\partial^2 x} + V(x,t)\Psi(\mathrm{x,t}) = \mathrm{i}\hbar\frac{\partial\psi(x,t)}{\partial t} \qquad (3-15)$$

Complex conjugate:

$$-\frac{\hbar^2}{2m}\frac{\partial^2\Psi^*(\mathrm{x,t})}{\partial^2 x} + V(x,t)\Psi^*(\mathrm{x,t}) = \mathrm{i}\hbar\frac{\partial\Psi^*(x,t)}{\partial t} \qquad (3-16)$$

By multiply the first by Ψ^* and the second equation by Ψ and subtract:

$$-\frac{\hbar^2}{2m}\left[\Psi^*\frac{\partial^2\Psi}{\partial^2 x}-\Psi\frac{\partial^2\Psi^*}{\partial^2 \mathrm{x}}\right]=i\hbar\left[\Psi^*\frac{\partial\Psi}{\partial t}+\Psi\frac{\partial\Psi^*}{\partial \mathrm{t}}\right]$$

$$-\frac{\hbar^2}{2m}\frac{\partial}{\partial x}\left[\Psi^*\frac{\partial\Psi}{\partial x}-\Psi\frac{\partial\Psi^*}{\partial \mathrm{x}}\right]=i\hbar\frac{\partial}{\partial t}(\Psi^*\Psi)$$

$$\therefore\ \frac{\partial}{\partial t}(\Psi^*\Psi)+\frac{\hbar}{2mi}\frac{\partial}{\partial x}\left(\Psi^*\frac{\partial\Psi}{\partial x}-\Psi\frac{\partial\Psi^*}{\partial x}\right)=0$$

$$\therefore\ \frac{\partial}{\partial t}P(x,t)+\frac{\partial}{\partial x}S_x=0 \qquad (3-17)$$

$$;\quad P=\Psi\Psi^*=|\Psi|^2 \qquad (3-18)$$

$$S_x=\frac{\hbar}{2mi}\left(\Psi^*\frac{\partial\Psi}{\partial x}-\Psi\frac{\partial\Psi^*}{\partial x}\right) \qquad (3-19)$$

Equation (3-17) is the Equation *of Quantity,* has the familiar form associated with the conservation of flow of a fluid of density P (eq. 3-18) and *probability density,* S_x (eq. 3-19).

3-4 The time independent Schrödinger Equation

If the potential V is a function of x alone, then the wave function $\Psi(x,t)$ can be written as a product of a function of x times a function of t, that is solutions of the form:

$$\Psi(x,t)=\Psi(x)\Phi(t) \qquad ; \qquad V(x,t)=V(x)$$

From Sch. Eq.

$$-\frac{\hbar^2}{2m}\Phi(t)\frac{\partial^2\Psi}{\partial^2 x}+V(x)\Psi\ (\mathrm{x})\Phi(\mathrm{t})=\mathrm{i}\hbar\Psi(\mathrm{x})\frac{\partial\Phi}{\partial t} \qquad (3-20)$$

$$-\frac{\hbar^2}{2m}\frac{1}{\Psi(x)}\frac{\partial^2\Psi}{\partial^2 x}+V(x)\Psi \quad = \mathrm{i}\hbar\frac{1}{\Phi}\frac{\partial\Phi}{\partial t} \qquad (3-21)$$

The left side of the equation dependence only on the variable x, the right side depends only on the variable t. since x and t independent variable. Both sides must be equal to a quantity which depends on neither x nor t in order that the equality of the left and right sides of the equation can be true for arbitrary values of the variables. Thus both sides must be equal to the same separation constant, C, that is:

$$\frac{1}{\Psi(x)}\left[-\frac{\hbar^2}{2m}\frac{\partial^2\Psi}{\partial^2 x}+V(x)\Psi\right]=\mathrm{C} \qquad (3-22)$$

$$i\hbar\frac{1}{\Phi(t)}\frac{d\Phi}{dt}=C \qquad (3-23)$$

$$\therefore \quad \Phi(t)=e^{-\frac{ict}{\hbar}}$$

$$\Phi(t), complex\ oscillatory\ function\ will\ freqancy\ \upsilon\ given\ by:$$

$$\frac{ct}{\hbar}\longrightarrow\ \omega t\Longrightarrow\omega=\frac{c}{\hbar}$$

$$\mathrm{c}=\hbar\upsilon \qquad\qquad \therefore\ C=E$$

$$-\frac{\hbar^2}{2m}\frac{\partial^2\Psi(\mathrm{x,t})}{\partial^2 x}+V(x,t)\Psi(\mathrm{x,t})=\mathrm{E}\,\Psi(x,t) \qquad (3-24)$$

$$\Psi(x,t)=e^{-i\frac{E}{\hbar}t}\ \Psi(x)$$

Where $\Psi(x)$ is called an *eigen function*, and equation (3-24) represent the Time Independent Schrödinger Equation (TISE).

Problems

1- for the distribution of gases in the example in section (3-1),

a) compute $\langle j^2 \rangle$ ***and*** $\langle j \rangle^2$.

b) Determine Δj for each j, and use equation (3-5) to compute the standard deviation.

c) use your result in (a) and (b) to check equation (3-6).

Problem 1.3 The needle on a broken car speedometer is free to swing, and bounces perfectly off the pins at either end, so that if you give it a flick it is equally likely to come to rest at any angle between 0 and π.

(a) What is the probability density, $\rho(\theta)$? [$\rho(\theta)\,d\theta$ is the probability that the needle will come to rest between θ and $(\theta + d\theta)$.] Graph $\rho(\theta)$ as a function of θ, from $-\pi/2$ to $3\pi/2$. (Of course, part of this interval is excluded, so ρ is zero there.) Make sure that the total probability is 1.

(b) Compute $\langle\theta\rangle$, $\langle\theta^2\rangle$, and σ for this distribution.

(c) Compute $\langle\sin\theta\rangle$, $\langle\cos\theta\rangle$, and $\langle\cos^2\theta\rangle$.

∗**Problem 1.6** Consider the **Gaussian** distribution

$$\rho(x) = Ae^{-\lambda(x-a)^2},$$

where A, a, and λ are constants. (Look up any integrals you need.)

(a) Use Equation 3-9 to determine A.

(b) Find $\langle x\rangle$, $\langle x^2\rangle$, and σ.

(c) Sketch the graph of $\rho(x)$.

Problem 1.7 At time $t = 0$ a particle is represented by the wave function

$$\Psi(x,0) = \begin{cases} Ax/a, & \text{if } 0 \le x \le a, \\ A(b-x)/(b-a), & \text{if } a \le x \le b, \\ 0, & \text{otherwise,} \end{cases}$$

where A, a, and b are constants.

(a) Normalize Ψ (that is, find A in terms of a and b).

(b) Sketch $\Psi(x, 0)$ as a function of x.

(c) Where is the particle most likely to be found, at $t = 0$?

(d) What is the probability of finding the particle to the left of a? Check your result in the limiting cases $b = a$ and $b = 2a$.

(e) What is the expectation value of x?

$*$**Problem 1.8** Consider the wave function

$$\Psi(x,t) = Ae^{-\lambda|x|}e^{-i\omega t},$$

where A, λ, and ω are positive real constants.

(a) Normalize Ψ.

(b) Determine the expectation values of x and x^2.

(c) Find the standard deviation of x. Sketch the graph of $|\Psi|^2$, as a function of x, and mark the points $(\langle x\rangle + \sigma)$ and $(\langle x\rangle - \sigma)$ to illustrate the sense in which σ represents the "spread" in x. What is the probability that the particle would be found outside this range?