

**EXAMPLE 5** To illustrate what is meant in Theorem 1, consider the experiment of tossing a single die. Let  $X$  denote the number of spots on the upper face:

$$f_X(x) = \left(\frac{1}{6}\right) I_{\{1, 2, \dots, 6\}}(x),$$

and

$$F_X(x) = \sum_{i=1}^5 (i/6) I_{[i, i+1)}(x) + I_{[6, \infty)}(x).$$

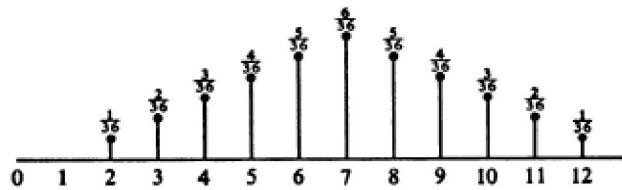


FIGURE 4

According to Theorem 1, for given  $f_X(\cdot)$ ,  $F_X(x)$  can be found for any  $x$ ; for instance, if  $x = 2.5$ ,

$$F_X(2.5) = \sum_{\{j: x_j \leq 2.5\}} f_X(x_j) = f_X(1) + f_X(2) = \frac{2}{6}.$$

And, if  $F_X(\cdot)$  is given,  $f_X(x)$  can be found for any  $x$ . For example, for  $x = 3$ ,

$$f_X(3) = F_X(3) - \lim_{0 < h \rightarrow 0} F_X(3 - h) = \left(\frac{3}{6}\right) - \left(\frac{2}{6}\right) = \frac{1}{6}. \quad \text{////}$$

The cumulative distribution function of a discrete random variable has steps at the mass points; that is, at the mass point  $x_j$ ,  $F_X(\cdot)$  has a step of size  $f_X(x_j)$ , and  $F_X(\cdot)$  is flat between mass points.

**Definition 6 Discrete density function** Any function  $f(\cdot)$  with domain the real line and counterdomain  $[0, 1]$  is defined to be a *discrete density function* if for some countable set  $x_1, x_2, \dots, x_n, \dots$ ,

- (i)  $f(x_j) > 0$  for  $j = 1, 2, \dots$
- (ii)  $f(x) = 0$  for  $x \neq x_j; j = 1, 2, \dots$
- (iii)  $\sum f(x_j) = 1$ , where the summation is over the points  $x_1, x_2, \dots, x_n, \dots$  ////

This definition allows us to speak of discrete density functions without reference to some random variable. Hence we can talk about properties that a density function might have without referring to a random variable.

### 3.2 Continuous Random Variables

**Definition 7 Continuous random variable** A random variable  $X$  is called *continuous* if there exists a function  $f_X(\cdot)$  such that  $F_X(x) = \int_{-\infty}^x f_X(u) du$  for every real number  $x$ . The cumulative distribution function  $F_X(\cdot)$  of a continuous random variable  $X$  is called *absolutely continuous*. ////

**Definition 8 Probability density function of a continuous random variable** If  $X$  is a continuous random variable, the function  $f_X(\cdot)$  in  $F_X(x) = \int_{-\infty}^x f_X(u) du$  is called the *probability density function* of  $X$ . ////

Other names that are used instead of probability density function include *density function*, *continuous density function*, and *integrating density function*.

Note that strictly speaking *the* probability density function  $f_X(\cdot)$  of a random variable  $X$  is not uniquely defined. All that the definition requires is that the integral of  $f_X(\cdot)$  gives  $F_X(x)$  for every  $x$ , and more than one function  $f_X(\cdot)$  may satisfy such requirement. For example, suppose  $F_X(x) = xI_{(0,1)}(x) + I_{(1,\infty)}(x)$ ; then  $f_X(u) = I_{(0,1)}(u)$  satisfies  $F_X(x) = \int_{-\infty}^x f_X(u) du$  for every  $x$ , and so  $f_X(\cdot)$  is a probability density function of  $X$ . However  $f_X(u) = I_{(0,1/2)}(u) + 69I_{(1/2,1)}(u) + I_{(1,1)}(u)$  also satisfies  $F_X(x) = \int_{-\infty}^x f_X(u) du$ . (The idea is that if the value of a function is changed at only a “few” points, then its integral is unchanged.) In practice a unique choice of  $f_X(\cdot)$  is often dictated by continuity considerations and for this reason we will usually allow ourselves the liberty of speaking of *the* probability density when in fact *a* probability density is more correct.

**Theorem 2** Let  $X$  be a continuous random variable. Then  $F_X(\cdot)$  can be obtained from an  $f_X(\cdot)$ , and vice versa.

telephone conversation. One could model this experiment by assuming that the distribution of  $X$  is given by  $F_X(x) = (1 - e^{-\lambda x})I_{(0,\infty)}(x)$ , where  $\lambda$  is some positive number. The corresponding probability density function would be given by  $f_X(x) = \lambda e^{-\lambda x}I_{(0,\infty)}(x)$ . If we assume that telephone conversations are measured in minutes,  $P[5 < X \leq 10] = \int_5^{10} \lambda e^{-\lambda x} dx = e^{-5\lambda} - e^{-10\lambda} = e^{-1} - e^{-2} \approx .23$  for  $\lambda = \frac{1}{5}$ , or  $P[5 < X \leq 10] = P[X \leq 10] - P[X \leq 5] = (1 - e^{-\lambda 10}) - (1 - e^{-\lambda 5}) = e^{-1} - e^{-2}$  for  $\lambda = \frac{1}{5}$ . ////

The probability density function is used to calculate the probability of events defined in terms of the corresponding continuous random variable  $X$ . For example,  $P[a < X \leq b] = \int_a^b f_X(x) dx$  for  $a < b$ .