Math 316-01 Intermediate Analysis

Questions for Section 29: The Riemann Integral

1. Some preliminaries: a partition of $[a, b]$ is $P = \{x_0, x_1, \ldots, x_n\}$, where $x_0 = a$, $x_n = b$, and $x_0 < x_1 < \cdots < x_n$. Given i, $1 \leq i \leq n$, we set $\Delta x_i = x_i - x_{i-1}$ (length of the subinterval $[x_{i-1}, x_i]$. A refinement of a partition P is a partition Q where $P \subseteq Q$. For example, $[a, b] = [1, 9]$, $P = \{1, 3, 4, 7, 9\}$, $\Delta_1 = 2$, $\Delta_2 = 1$, $\Delta_3 = 3$, $\Delta_4 = 2$, $Q = \{1, 2, 3, 4, 7, 8, 9\}$.

2. Given a function $f : [a, b] \to \mathbb{R}$ such that $f([a, b])$ is bounded, and given a partition $P = \{x_0, x_1, \cdots, x_n\}$ of $[a, b]$, we set $M_i = \sup f([x_{i-1}, x_i])$ and $m_i = \inf f([x_{i-1}, x_i]$. For example, if $f : [1, 9] \to \mathbb{R}$ is given by $f(x) = x^2$, and if $P = \{1, 3, 4, 7, 9\}$, then $M_1 = 9$, $M_2 = 16$, $M_3 = 49$, $M_4 = 81$, $m_1 = 1$, $m_2 = 9$, $m_3 = 16$, $m_4 = 49$.

3. Given a bounded function $f : [a, b] \to \mathbb{R}$ and a partition $P = \{x_0, x_1, \ldots, x_n\}$ of $[a, b]$, we set $U(f, P) = M_1\Delta_1 + \cdots + M_n\Delta_n$ (the upper sum) and $L(f, P) = m_1\Delta_1 + \cdots + m_n\Delta_n$ (the lower sum). We always have $L(f, P) \leq U(f, P)$. In our example above, $U(f, P) = 9 \cdot 2 + 16 \cdot 1 + 49 \cdot 3 + 81 \cdot 2 = 343$ and $L(f, P) = 1 \cdot 2 + 9 \cdot 1 + 16 \cdot 3 + 49 \cdot 2 = 157$.

4. If $f : [a, b] \to \mathbb{R}$ is a bounded function and P and Q are partitions of $[a, b]$ such that $P \subseteq Q$, then $U(f, P) \geq U(f, Q)$.

Proof: Q is obtained by adding partition points to P. We will prove the result assuming that Q contains one more point than P. So consider $Q = P \cup \{y\}$ where $x_{i-1} < y < x_i$. The only difference between $U(f, P)$ and $U(f, Q)$ is that the term $\sup f([x_{i-1}, x_i])(x_i - x_{i-1})$ in $U(f, P)$ is replaced by $\sup f([x_{i-1}, y])(y - x_{i-1}) + \sup f([y, x_i])(x_i - y)$ in $U(f, Q)$. However, $\sup f([x_{i-1}, x_i]) \geq \sup f([x_{i-1}, y])$ and $\sup f([x_{i-1}, x_i]) \geq \sup f([y, x_i])$, therefore

$$\sup f([x_{i-1}, x_i])(x_i - x_{i-1}) = \sup f([x_{i-1}, x_i])(y - x_{i-1}) + \sup f([x_{i-1}, x_i])(x_i - y)$$

$$\geq \sup f([x_{i-1}, y])(y - x_{i-1}) + \sup f([y, x_i])(x_i - y).$$

Since one term in $U(f, P)$ is replaced by a smaller sum of two terms in $U(f, Q)$, we must have $U(f, P) \geq U(f, Q)$.

5. If $f : [a, b] \to \mathbb{R}$ is a bounded function and P and Q are partitions of $[a, b]$ such that $P \subseteq Q$, then $L(f, P) \geq L(f, Q)$.

Proof: The proof is similar that that above. The infimum of f over $[x_{i-1}, x_i]$ is $\leq$ the infimum of f over $[x_{i-1}, y]$ and over $[y, x_i]$.

6. If $f : [a, b] \to \mathbb{R}$ is a bounded function and P and Q are arbitrary partitions of $[a, b]$, then $L(f, P) \leq U(f, Q)$.

Proof: Note that $P \cup Q$ is a refinement of P and a refinement of Q. So we have $L(f, P) \leq L(f, P \cup Q) \leq U(f, P \cup Q) \leq U(f, Q)$, combining the results in comments 4, and 5 above.

7. Let $f : [a, b] \to \mathbb{R}$ be a bounded function. We can see that the set $\{U(f, P) : P \vdash [a, b]\}$ is bounded below by every $L(f, Q)$. Set $U(f) = \inf\{U(f, P) : P \vdash [a, b]\}$. We can also see that the set $\{L(f, Q) : Q \vdash [a, b]\}$ is bounded above by every $U(f, P)$. Set $L(f) = \sup\{L(f, Q) : Q \vdash [a, b]\}$. Then we have $L(f, P) \leq U(f)$ for all P, therefore $L(f) \leq U(f)$. When $L(f) < U(f)$ then we say that f is not integrable over $[a, b]$. But when $L(f) = U(f)$ then we say that f is integrable over $[a, b]$, and we define

$$\int_a^b f = L(f) = U(f).$$

8. An example of a non-integrable function $g : [0, 2] \to \mathbb{R}$ is given in Example 29.8, page 273. We have $L(g) = 0$, $U(g) = 2$.

9. We will comment on Example 29.7, page 272. Let $f : [0, 1] \to \mathbb{R}$ be defined by $f(x) = x^2$. Then $L(f) = U(f) = \frac{1}{3}$. To see this, let P_n denote the partition $\{0, \frac{1}{n}, \frac{2}{n}, \ldots, 1\}$. Then

$$L(f, P_n) = \frac{1}{3}\left(\frac{n-1}{n}\right)\left(\frac{2n-1}{2n}\right)$$

and

$$U(f, P_n) = \frac{1}{3}\left(\frac{n+1}{n}\right)\left(\frac{2n+1}{2n}\right).$$

We

$$L(f) = \sup\{L(f, P) : P \vdash [0, 1]\} \geq \sup\{L(f, P_n) : n \in \mathbb{N}\} = \frac{1}{3}$$

and

$$U(f) = \inf\{U(f, P) : P \vdash [0, 1]\} \leq \inf\{U(f, P_n) : n \in \mathbb{N}\} = \frac{1}{3},$$

therefore $L(f) \geq U(f)$. But we always have $L(f) \leq U(f)$, therefore $L(f) = U(f)$. Moreover $\frac{1}{3} \leq L(f) \leq U(f) \leq \frac{1}{3}$, which can only occur of $L(f) = U(f) = \frac{1}{3}$. Therefore $\int_0^1 f = \frac{1}{3}$.

10. We will comment on the proof of Theorem 29.9. This theorem is to be interpreted as an alternative definition of integrability, which should be useful for proofs in later sections. Assume $f : [a, b] \to \mathbb{R}$ is bounded and integrable on $[a, b]$. Then $L(f) = U(f) = \int_a^b f$. Let $\epsilon > 0$ be given. Then there must exist a partition P such that $U(f, P) - L(f, P) < \epsilon$. To see this choose a partition P_1 such that $L(f) - \frac{\epsilon}{2} < L(f, P_1) \leq L(f)$ and choose a partition P_2 such that $U(f) \leq U(f, P_2) < U(f) + \frac{\epsilon}{2}$. Then we have

$$L(f) - \frac{\epsilon}{2} < L(f, P_1 \cup P_2) \leq U(f, P_1 \cup P_2) < U(f) + \frac{\epsilon}{2},$$

which implies that $L(f, P_1 \cup P_2)$ and $U(f, P_1 \cup P_2)$ are trapped between $\int_a^b f - \frac{\epsilon}{2}$ and $\int_a^b f + \frac{\epsilon}{2}$. This means that the gap between $L(f, P_1 \cup P_2)$ and $U(f, P_1 \cup P_2)$ is smaller than ϵ. Hence integrability implies we can find a partition P such that $U(f, P) - L(f, P) < \epsilon$.

Conversely, suppose that $f : [a, b] \to \mathbb{R}$ is a bounded function that meets this criterion. We will prove that f is integrable over $[a, b]$. For all $n \in \mathbb{N}$ there exists P_n such that $0 \leq U(f, P_n) - L(f, P_n) < \frac{1}{n}$. This implies $0 \leq U(f) - L(f, P_n) < \frac{1}{n}$. Hence $\lim L(f, P_n) = U(f)$. We also have $0 \leq U(f, P_n) - L(f) < \frac{1}{n}$. This implies $\lim U(f, P_n) = L(f)$. We also have $\lim\ (U(f, P_n) - L(f, P_n)) = 0$. Using limit properties, this implies $U(f) - L(f) = 0$. Therefore $L(f) = U(f)$ and f is integrable.

Homework for Section 29, due ??? (only the starred problems will be graded):

1, 2, 7*, 8*, 9*, 13*, 16*, 29*

Hints:

7. Mimic Example 29.7, page 272 and Comment 9 of these notes. Use $1^3 + 2^3 + \cdots + n^3 = \frac{1}{4}n^2(n+1)^2$.

8. Let $f : [0, 1] \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ -1 & x \notin \mathbb{Q} \end{cases}.$$

Show that $L(f) < U(f)$ as in Example 29.8, page 273. Then show that $f^2 : [0,1] \to \mathbb{R}$ defined by $f^2(x) = f(x)^2 = 1$ is integrable with $L(f) = U(f) = 1$.

9. You should be able to construct a counterexample using $h : [0,1] \to \mathbb{Q}$ defined by

$$h(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ -1 & x \notin \mathbb{Q} \end{cases}.$$

Compare with Example 29.8, page 273.

13. Prove one case of the contrapositive, namely that if $f(c) > 0$ for some $c \in [a,b]$ then $L(f) > 0$. Note that by continuity of f at c there exists a $\delta > 0$ such that $x \in [a,b]$ and $c - \delta < x < c + \delta$ implies $|f(x) - f(c)| < |f(c)|$, which implies $f(x) > 0$ for these values of x. Now construct a partition P which takes advantage of this fact, so that $L(f,P) > 0$. Be specific about the contents of P. This implies $L(f) \geq L(f,P) > 0$. It will help to draw a diagram first.

16. To make the problem more concrete and manageable, assume that $f : [0,10] \to \mathbb{R}$ is defined by

$$f(x) = \begin{cases} 100 & x = 2 \\ 200 & x = 5 \\ 0 & x \in [0,10] \backslash \{2,5\}. \end{cases}$$

It should be clear that $L(f) \geq 0$. So it will suffice to show that $U(f) = 0$. This will imply that $U(f) \leq L(f)$, hence $L(f) = U(f) = 0$, which implies that f is integrable over $[0,10]$ and $\int_0^{10} f = 0$. To show that $U(f) = 0$, show that for all $\epsilon > 0$ there exists a partition P such that $U(f,P) < \epsilon$. Then $U(f) = \inf \{U(f,P) : P \vdash [0,10]\} = 0$. You should construct the partition in such a way that $2 \in [x_1, x_2]$, $5 \in [x_3, x_4]$, and the size of these intervals is small enough to force $U(f,P) < \epsilon$. It will help to draw a diagram first.