3 Sequences

3.1 Basic Properties

Definition 3.1. A *sequence* is a function $a : \mathbb{N} \to \mathbb{R}$.

Instead of using the standard function notation of $a(n)$ for sequences, it is usually more convenient to write the argument of the function as a subscript, a_n.

Example 3.1. Let the sequence $a_n = 1 - 1/n$. Then an easy calculation shows $a_1 = 0,\ a_2 = 1/2,\ a_3 = 2/3$, etc.

Example 3.2. Let the sequence $b_n = 2^n$. It's easy to see $b_1 = 2,\ b_2 = 4,\ b_3 = 8$, etc.

Definition 3.2. A sequence a_n is *bounded* if $\{a_n : n \in \mathbb{N}\}$ is a bounded set. This definition is extended in the obvious way to *bounded above* and *bounded below*.

The sequence of Example 3.1 is bounded, but the sequence of Example 3.2 is not.

Definition 3.3. A sequence a_n *converges* to $L \in \mathbb{R}$ if for all $\varepsilon > 0$ there exists an $N \in \mathbb{N}$ such that whenever $n \geq N$, then $|a_n - L| < \varepsilon$. If a sequence does not converge, then it is said to *diverge*.

When a_n converges to L, we write $\lim_{n\to\infty} a_n = L$, or often, more simply, $a_n \to L$.

Example 3.3. Let a_n be as in Example 3.1. We claim $a_n \to 1$. To see this, let $\varepsilon > 0$ and choose $N \in \mathbb{N}$ such that $1/N < \varepsilon$. Then, if $n \geq N$

$$|a_n - 1| = |(1 - 1/n) - 1| = 1/n \leq 1/N < \varepsilon,$$

so $a_n \to 1$.

The sequence b_n of Example 3.2 diverges. To see this, suppose not. Then there is an $L \in \mathbb{R}$ such that $b_n \to L$. If $\varepsilon = 1$, there must be an $N \in \mathbb{N}$ such that $|b_n - L| < \varepsilon$ whenever $n \geq N$. Choose $n \geq N$. $|L - 2^n| < 1$ implies $L < 2^n + 1$. But, then

$$b_{n+1} - L = 2^{n+1} - L > 2^{n+1} - (2^n + 1) = 2^n - 1 \geq 1 = \varepsilon.$$

This violates the condition on N. We conclude that for every $L \in \mathbb{R}$ there exists an $\varepsilon > 0$ such that for no $N \in \mathbb{N}$ is it true that whenever $n \geq N$, then $|b_n - L| < \varepsilon$. Therefore, b_n diverges.

Definition 3.4. A sequence a_n *diverges to* ∞ if for every $B > 0$ there is an $N \in \mathbb{N}$ such that $n \geq N$ implies $a_n > B$. The sequence a_n is said to *diverge to* $-\infty$ if $-a_n$ diverges to ∞.

When a_n diverges to ∞, we write $\lim_{n\to\infty} a_n = \infty$, or often, more simply, $a_n \to \infty$.

Example 3.4. It is easy to prove that the sequence of Example 3.2 diverges to ∞.

Theorem 3.1. *If $a_n \to L$, then L is unique.*

Proof. Suppose $a_n \to L_1$ and $a_n \to L_2$. Let $\varepsilon > 0$. According to Definition 3.2, there exist $N_1, N_2 \in \mathbb{N}$ such that $n \geq N_1$ implies $|a_n - L_1| < \varepsilon/2$ and $n \geq N_2$ implies $|a_n - L_2| < \varepsilon/2$. Set $N = \max\{N_1, N_2\}$. If $n \geq N$, then

$$|L_1 - L_2| = |L_1 - a_n + a_n - L_2| \leq |L_1 - a_n| + |a_n - L_2| < \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

Since ε is an arbitrary positive number, this implies $L_1 = L_2$. $\square$

Theorem 3.2. *$a_n \to L$ iff for all $\varepsilon > 0$, the set $\{n : a_n \notin (L - \varepsilon, L + \varepsilon)\}$ is finite.*

Proof. ($\Rightarrow$) Let $\varepsilon > 0$. According to Definition 3.2, there is an $N \in \mathbb{N}$ such that $\{a_n : n \geq N\} \subset (L-\varepsilon, L+\varepsilon)$. Then $\{n : a_n \notin (L-\varepsilon, L+\varepsilon)\} \subset \{1, 2, \ldots, N-1\}$.

($\Leftarrow$) Let $\varepsilon > 0$. By assumption $\{n : a_n \notin (L - \varepsilon, L + \varepsilon)\}$ is finite, so let $N = \max\{n : a_n \notin (L - \varepsilon, L + \varepsilon)\} + 1$. If $n \geq N$, then $a_n \in (L - \varepsilon, L + \varepsilon)$, so, by Definition 3.2, $a_n \to L$. $\square$

Corollary 3.3. *If a_n converges, then a_n is bounded.*

Proof. Suppose $a_n \to L$. According to Theorem 3.2 there are a finite number of terms of the sequence lying outside $(L - 1, L + 1)$. Since any finite set is bounded, the conclusion is obvious. $\square$

Theorem 3.4. *Let a_n and b_n be sequences such that $a_n \to A$ and $b_n \to B$. Then*

(a) $a_n + b_n \to A + B$,

(b) $ca_n \to cA$, *for all* $c \in \mathbb{R}$,

(c) $a_n b_n \to AB$, *and*

(d) $a_n/b_n \to A/B$ *as long as* $b_n \neq 0$ *for all* $n \in \mathbb{N}$ *and* $B \neq 0$.

Proof. (a) Let $\varepsilon > 0$. There are $N_1, N_2 \in \mathbb{N}$ such that $n \geq N_1$ implies $|a_n - A| < \varepsilon/2$ and $n \geq N_2$ implies $|b_n - B| < \varepsilon/2$. Define $N = \max\{N_1, N_2\}$. If $n \geq N$, then

$$|(a_n + b_n) - (A + B)| \leq |a_n - A| + |b_n - B| < \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

Therefore $a_n + b_n \to A + B$.

(b) If $c = 0$, the statement is obvious. So, assume $c \neq 0$ and let $\varepsilon > 0$. Choose $N \in \mathbb{N}$ so that whenever $n \geq N$, then $|a_n - A| < \varepsilon/|c|$. If $n \geq \mathbb{N}$, then

$$|ca_n - cA| = |c||a_n - A| < |c|\varepsilon/c = \varepsilon.$$

Therefore, $ca_n \to cA$.

(c) Let $\varepsilon > 0$ and $\alpha > 0$ be an upper bound for $|a_n|$. Choose $N_1, N_2 \in \mathbb{N}$ such that $n \geq N_1 \implies |a_n - A| < \varepsilon/2(|B|+1)$ and $n \geq N_2 \implies |b_n - B| < \varepsilon/2\alpha$. If $n \geq N = \max\{N_1, N_2\}$, then

$$\begin{aligned}|a_n b_n - AB| &= |a_n b_n - a_n B + a_n B - AB| \\ &\leq |a_n b_n - a_n B| + |a_n B - AB| \\ &= |a_n||b_n - B| + ||B||a_n - A| \\ &< \alpha \frac{\varepsilon}{2\alpha} + |B| \frac{\varepsilon}{2(|B|+1)} \\ &< \varepsilon/2 + \varepsilon/2 = \varepsilon.\end{aligned}$$

(d) First, notice that it suffices to show that $1/b_n \to B$, because part (c) of this theorem can be used to achieve the full result.

Let $\varepsilon > 0$. Choose $N \in \mathbb{N}$ so that $n \geq N \implies |b_n| > B/2$ and $|b_n - B| < B^2\varepsilon/2$. Then, when $n \geq N$,

$$\left| \frac{1}{b_n} - \frac{1}{B} \right| = \left| \frac{B - b_n}{b_n B} \right| < \left| \frac{B^2\varepsilon/2}{(B/2)B} \right| = \varepsilon.$$

Therefore $1/b_n \to 1/B$.

□

Theorem 3.5 (Sandwich Theorem). *Suppose a_n, b_n and c_n are sequences such that $a_n \leq b_n \leq c_n$ for all $n \in \mathbb{N}$.*

(a) If $a_n \to L$ and $c_n \to L$, then $b_n \to L$.

(b) If $b_n \to \infty$, then $c_n \to \infty$.

(c) If $c_n \to -\infty$, then $b_n \to -\infty$.

Proof. (a) Let $\varepsilon > 0$. There is an $N \in \mathbb{N}$ large enough so that when $n \geq N$, then $L - \varepsilon < a_n$ and $c_n < L + \varepsilon$. These inequalities imply $L - \varepsilon < a_n \leq b_n \leq c_n < L + \varepsilon$. Therefore, $c_n \to L$.

(b) Let $B > 0$ and choose $N \in \mathbb{N}$ so that $n \geq N \implies b_n > B$. Then $c_n \geq b_n > B$ whenever $n \geq N$. This shows $c_n \to \infty$.

(c) This is essentially the same as part (b).

□

Problem 12. Show that the sequence $a_n = \dfrac{3n+1}{2n+3}$ converges.

Extra Credit 3. If $a_n \to L$, then what can you say about

$$\sigma_n = \frac{a_1 + a_2 + \cdots + a_n}{n}?$$

Is there a divergent sequence a_n such that σ_n converges?

Problem 13. A sequence a_n converges to 0 iff $|a_n|$ converges to 0.

3.2 Monotone Sequences

Definition 3.5. A sequence a_n is *increasing*, if $a_{n+1} \geq a_n$ for all $n \in \mathbb{N}$. It is *strictly increasing* if $a_{n+1} > a_n$ for all $n \in \mathbb{N}$.

A sequence a_n is *decreasing*, if $a_{n+1} \leq a_n$ for all $n \in \mathbb{N}$. It is *strictly decreasing* if $a_{n+1} < a_n$ for all $n \in \mathbb{N}$.

If a_n is any of the four types listed above, then it is said to be a *monotone* sequence.

Theorem 3.6. *A bounded monotone sequence converges.*

Proof. Suppose a_n is a bounded increasing sequence, $L = \text{lub}\,\{a_n : n \in \mathbb{N}\}$ and $\varepsilon > 0$. Clearly, $a_n \leq L$ for all $n \in \mathbb{N}$. According to Theorem 2.10, there exists an $N \in \mathbb{N}$ such that $a_N > L - \varepsilon$. Then $L \geq a_n \geq a_N > L - \varepsilon$ for all $n \geq N$. This shows $a_n \to L$.

If a_n is decreasing, let $b_n = -a_n$ and apply the preceding argument. □

Theorem 3.7. *An unbounded monotone sequence diverges to* ∞ *or* $-\infty$*, depending on whether it is increasing or decreasing, respectively.*

Proof. Suppose a_n is increasing and unbounded. If $B > 0$, the fact that a_n is unbounded yields an $N \in \mathbb{N}$ such that $a_N > B$. Since a_n is increasing, $a_n \geq a_N > B$ for all $n \geq N$. This shows $a_n \to \infty$.

The proof when the sequence decreases is similar. □

3.3 The Nested Interval Theorem

Definition 3.6. A collection of sets $\{S_n : n \in \mathbb{N}\}$ is said to be *nested*, if $S_{n+1} \subset S_n$ for all $n \in \mathbb{N}$.

Theorem 3.8 (Nested Interval Theorem). *If* $I_n = [a_n, b_n]$ *is a nested collection of closed intervals such that* $\lim_{n\to\infty} b_n - a_n = 0$*, then there is an* $x \in \mathbb{R}$ *such that* $\bigcap_{n\in\mathbb{N}} I_n = \{x\}$*.*

Proof. Since the intervals are nested, it's clear that a_n is an increasing sequence bounded above by b_1 and b_n is a decreasing sequence bounded below by a_1. Applying Theorem 3.6 twice, we find there are $\alpha, \beta \in \mathbb{R}$ such that $a_n \to \alpha$ and $b_n \to \beta$.

We claim $\alpha = \beta$. To see this, let $\varepsilon > 0$ and use the "shrinking" condition on the intervals to pick $N \in \mathbb{N}$ so that $b_N - a_N < \varepsilon$. The nestedness of the intervals implies $a_N \leq a_n < b_n \leq b_N$ for all $n \geq N$. Therefore

$$a_N \leq \text{lub}\,\{a_n : n \geq N\} = \alpha \leq b_N \text{ and } a_N \leq \text{glb}\,\{b_n : n \geq N\} = \beta \leq b_N.$$

This shows $|\alpha - \beta| \leq |b_N - a_N| < \varepsilon$. Since $\varepsilon > 0$ was chosen arbitrarily, we conclude $\alpha = \beta$.

Let $x = \alpha = \beta$. It remains to show that $\bigcap_{n\in\mathbb{N}} I_n = \{x\}$.

First, we shaw that $x \in \bigcap_{n\in\mathbb{N}} I_n$. To do this, fix $N \in \mathbb{N}$. Since a_n increases to x, it's clear that $x \geq a_N$. Similarly, $x \leq b_N$. Therefore $x \in [a_N, b_N]$. Because N was chosen arbitrarily, it follows that $x \in \bigcap_{n\in\mathbb{N}} I_n$.

Next, suppose there are $x, y \in \bigcap_{n\in\mathbb{N}} I_n$ and let $\varepsilon > 0$. Choose $N \in \mathbb{N}$ such that $b_N - a_N < \varepsilon$. Then $\{x, y\} \subset \bigcap_{n\in\mathbb{N}} I_n \subset [a_N, b_N]$ implies $|x - y| < \varepsilon$. Since ε was chosen arbitrarily, we see $x = y$. Therefore $\bigcap_{n\in\mathbb{N}} I_n = \{x\}$. □

Example 3.5. If $I_n = (0, 1/n]$ for all $n \in \mathbb{N}$, then the collection $\{I_n : n \in \mathbb{N}\}$ is nested, but $\bigcap_{n\in\mathbb{N}} I_n = \emptyset$. This shows the assumption that the intervals be closed in the Nested Interval Theorem is necessary.

Example 3.6. If $I_n = [n, \infty)$ then the collection $\{I_n : n \in \mathbb{N}\}$ is nested, but $\bigcap_{n\in\mathbb{N}} I_n = \emptyset$. This shows that the assumption that the lengths of the intervals be bounded is necessary.

Extra Credit 4. If a_n is a sequence such that $\frac{a_n-1}{a_n+1} \to 0$, then does $\lim_{n\to\infty} a_n$ exist?

Extra Credit 5. Suppose a sequence is defined by $a_1 = 0$, $a_1 = 1$ and $a_{n+1} = \frac{1}{2}(a_n + a_{n-1})$ for $n \geq 2$. Prove a_n converges, and determine its limit.

Problem 14. Prove that the sequence $a_n = n^3/n!$ converges.

3.4 Subsequences

Definition 3.7. Let a_n be a sequence and $\sigma : \mathbb{N} \to \mathbb{N}$ be a strictly increasing function. Then $b_n = a_{\sigma(n)}$ is a subsequence of a_n.

The idea here is that the subsequence b_n is a new sequence formed from an old sequence a_n by possibly leaving terms out of a_n. In other words, we see that all the terms of b_n must also appear in a_n, and they must appear in the same order.

Example 3.7. If $a_n = \sin(n\pi/2)$, then some possible subsequences are

$$b_n = a_{2n-1} \implies b_n = (-1)^{n+1},$$

$$c_n = a_{2n} \implies c_n = 0,$$

and

$$d_n = a_{n^2} \implies d_n = (1 + (-1)^{n+1})/2.$$

Theorem 3.9. *$a_n \to L$ iff every subsequence of a_n converges to L.*

Proof. ($\Rightarrow$) Suppose $\sigma : \mathbb{N} \to \mathbb{N}$ is strictly increasing, as in the preceding definition. Clearly, $\sigma(1) \geq 1$. Suppose $\sigma(n) \geq n$ for some $n \in \mathbb{N}$. Then $\sigma(n+1) > \sigma(n) \geq n \Rightarrow \sigma(n+1) \geq n+1$. This simple induction argument has established $\sigma(n) \geq n$ for all $n \in \mathbb{N}$.

Now, suppose $a_n \to L$ and $b_n = a_{\sigma(n)}$ is a subsequence of a_n. If $\varepsilon > 0$, there is an $N \in \mathbb{N}$ such that $n \geq N$ implies $a_n \in (L - \varepsilon, L + \varepsilon)$. From the preceding paragraph, it follows that when $n \geq N$, then $b_n = a_{\sigma(n)} = a_m$ for some $m \geq n$. So, $b_n \in (L - \varepsilon, L + \varepsilon)$ and $b_n \to L$.

($\Leftarrow$) Since a_n is a subsequence of itself, it is obvious that $a_n \to L$. □

Any sequence has an uncountable number of subsequences. Even if the original sequence diverges, it is possible there are convergent subsequences. For example, consider the divergent sequence $a_n = (-1)^n$. In this case, a_n diverges, but the two subsequences a_{2n} and a_{2n+1} are constant sequences, so they converge.

Problem 15. If a_n is a sequence such that every subsequence of a_n has a further subsequence converging to 0, then $a_n \to 0$.