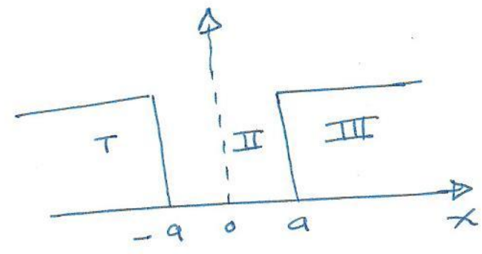
Square well - potential

A particle in a square well ($E < V_0$)

$$V(x) = \begin{cases} V_0 & x \leq a \\ 0 & -a \leq x \leq a \\ V_0 & x \geq a \end{cases}$$



write the Schrödenger equation in three region and solve for the wave function :

In the first region $\psi_1 = A e^{\alpha x}$

" " second " $\psi_2 = B e^{ikx} + C e^{-ikx}$

" " third " $\psi_3 = D e^{-\alpha x}$

where $\alpha = \sqrt{\frac{2m(V_0 - E)}{\hbar^2}}$; $k = \sqrt{\frac{2mE}{\hbar^2}}$

ملاحظة/ حيث توجد الدوال الموجية الثلاثة اعلاه بنفس الطرق المتبعة في حالة عتبة الجهد وحاجز الجهد.

ولإيجاد قيم الثوابت A, B, C, D علينا تطبيق الشروط الحدودية "Boundary Condition" عند $x=-a$ و $x=a$.

Boundary Condition: general case

at $x = -a$ $\quad \psi_1 = \psi_2 \Big|_{x=-a}$ & $\dfrac{\partial \psi_1}{\partial x} = \dfrac{\partial \psi_2}{\partial x}\Big|_{x=-a}$

$$\psi_1 = A e^{\alpha x} \quad , \quad \psi_2 = B e^{ikx} + C e^{-ikx}$$

$$\psi_3 = D e^{-\alpha x}$$

The first Condition:

$$\psi_1 = \psi_2 \Big|_{x=-a}$$

$$A e^{\alpha x} = B e^{ikx} + C e^{-ikx}$$

$$\psi_1 = \psi_2 \Big|_{x=-a}$$

$$A e^{-\alpha a} = B e^{-ika} + C e^{ika} \quad \text{—} (1)$$

The second Condition:

$$\left| \frac{\partial \psi_1}{\partial x} = \frac{\partial \psi_2}{\partial x} \right|_{x=-a}$$

$$\alpha A e^{\alpha x} = ik B e^{ikx} - ik C e^{-ikx}$$

$$\frac{\partial \psi_1}{\partial x}\Big|_{x=-a} = \frac{\partial \psi_2}{\partial x}\Big|_{x=-a}$$

$$\alpha A e^{-\alpha a} = ik B e^{-ika} - ik C e^{ika} \quad \text{—} (2)$$

بالمقارنة بين ① و ② نلاحظ ان الطرف الايسر من المعادلتين
متساوي والفرق فقط بـ (α) في معادلة (2)

$$\alpha B e^{-ika} + \alpha C e^{ika} = ik B e^{-ika} - ik C e^{ika}$$

نضرب معادلة رقم ① بـ α وطرح المعادلتين ① و ② بعد الضرب

$$\alpha B e^{-ika} - ikB e^{-ika} = -ikC e^{ika} - \alpha C e^{ika}$$

$$(\alpha - ik) B e^{-ika} = -(\alpha + ik) C e^{ika}$$

$$\therefore \frac{B}{C} = \frac{-(\alpha + ik)}{(\alpha - ik)} \frac{e^{ika}}{e^{-ika}} \qquad \left(\frac{1}{e^{-ika}} = e^{ika} \right)$$

$$\boxed{\therefore \frac{B}{C} = \frac{-(\alpha + ik)}{(\alpha - ik)} e^{2ika}} \quad \text{---} *$$

at $x = a$, $\psi_2 = \psi_3 \big|_{x=a}$ & $\frac{\partial \psi_2}{\partial x} = \frac{\partial \psi_3}{\partial x} \Big|_{x=a}$

$$\psi_2 = B e^{ikx} + C e^{-ikx} \quad , \quad \psi_3 = D e^{-\alpha x}$$

The first Condition:

$$\psi_2 = \psi_3 \big|_{x=a}$$

$$B e^{ika} + C e^{-ika} = D e^{-\alpha a} \quad \text{---} ①$$

The second Condition:

$$\frac{\partial \psi_2}{\partial x}\Big|_{x=a} = \frac{\partial \psi_3}{\partial x}\Big|_{x=a}$$

$$ikB e^{ika} - ikC e^{-ika} = -\alpha D e^{-\alpha a} \quad \text{---} ②$$

من مقارنة المعادلتين اعلاه ينتج :

$$ikB e^{ika} - ikC e^{-ika} = -\alpha B e^{ika} - \alpha C e^{-ika}$$

$$ikB e^{ika} + \alpha B e^{ika} = +ikC e^{-ika} - \alpha C e^{-ika}$$

$$(ik + \alpha) B e^{ika} = -(\alpha - ik) C e^{-ika}$$

$$(\alpha+ik)Be^{-ika} = -(\alpha-ik)Ce^{ika}$$

$$\therefore \boxed{\frac{B}{C} = -\frac{(\alpha-ik)}{(\alpha+ik)} \cdot e^{2ika}} \sim\sim (**)$$

وبتعويض معادلة (*) في معادلة (**) ينتج :

$$B^2/C^2 = 1 \Rightarrow B = \pm C$$

وبتعويض قيمة الثابت B في معادلة ψ_2 ينتج :

$$\psi_2 = Be^{ikx} + Ce^{-ikx}$$

$$\psi_2(x) = C(e^{ikx} \pm e^{-ikx})$$

either $\psi_2(x) = C(e^{ikx} + e^{-ikx}) = C\cos kx$

or $\psi_2(x) = C(e^{ikx} - e^{-ikx}) = C\sin kx$

even Solution: $\psi(-x) = \psi(x)$

odd Solution: $\psi(-x) \quad -\psi(x)$

even parity (solution): at $\boxed{B=+C}$

from equation * $\Rightarrow -(\alpha+ik)e^{2ika} = \alpha - ik$

$$\Rightarrow ik(1-e^{2ika}) = \alpha(1+e^{2ika})$$ نسحب e^{ika} خارج الأقواس

$$ik\,e^{ika}(e^{-ika} - e^{ika}) = \alpha\, e^{ika}(e^{-ika} + e^{ika})$$

$$2k\left(\frac{e^{-ika}-e^{ika}}{2i}\right) = 2\alpha\left(\frac{e^{-ika}+e^{ika}}{2}\right)$$

$$k\,\sin(ka) = \alpha\,\cos(ka)$$

$$k\tan ka = \alpha$$

$$ka\tan ka = \alpha a \quad \text{— } *$$

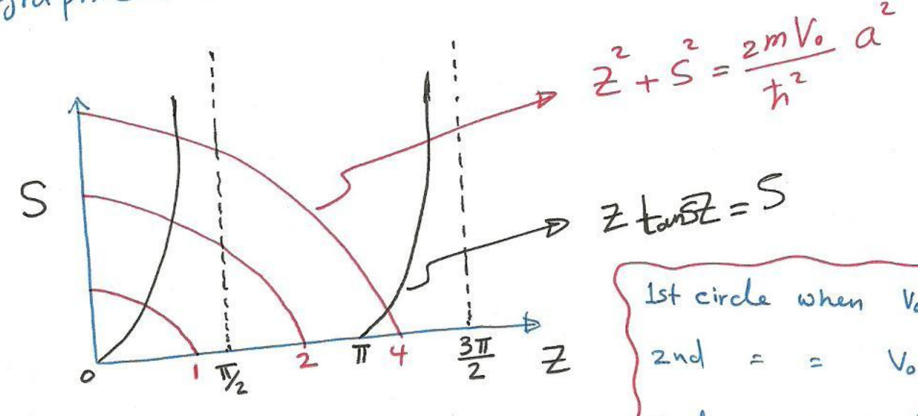
let $ka = Z$, $\alpha a = S$

معادلة (*) تصبح كالآتي

$$(**) \text{ —} \begin{cases} Z\tan Z = S \\ Z^2 + S^2 = \dfrac{2m}{\hbar^2} V_0 a^2 \end{cases} \rightsquigarrow \; Z = \overbrace{\sqrt{\frac{2mE}{\hbar^2}}}^{k}\, a \quad \text{and} \quad S = \underbrace{\sqrt{\frac{2m}{\hbar^2}(V_0 - E)}}_{\alpha}\, a$$

Energy Level مستويات الطاقة

The energy Level are bound from a numerical or graphical solution of equation (**)



1st circle when $V_0 a^2 = \frac{\hbar^2}{2m}$

2nd " " $V_0 a^2 = \frac{4^2\hbar^2}{2m}$

3rd " " $V_0 a^2 = \frac{16\hbar^2}{2m}$

ii) odd parity solution / فيصل

$$\frac{B}{C} = -1 \quad \text{or} \quad B = -C$$

from equation ( )

$$(\alpha + ik)e^{2ika} = \alpha - ik$$

$$ik(1 + e^{2ika}) = \alpha(1 - e^{2ika})$$

$$ik\, e^{ika}(e^{-ika} + e^{ika}) = \alpha\, e^{ika}(e^{-ika} - e^{ika})$$

$$2k\left(\frac{e^{ika} + e^{-ika}}{2}\right) = 2\alpha\left(\frac{e^{ika} - e^{-ika}}{2}\right)$$

$$k \cos ka = -\alpha \sin ka$$

$$k \cot ka = -\alpha \implies ka \cot ka = -\alpha a$$

$$\therefore \boxed{Z \cot Z = -S}$$

and $\boxed{Z^2 + S^2 = \frac{2m}{\hbar^2} V_0 a^2}$ —— (***)