Chapter 2

Determinants

The determinant of a matrix can be thought of as a function which associates a real number with every square matrix. In this chapter, we will see how this association is made. We will also see the kind of information about a matrix its determinant can give us.

2.1 Introduction to Determinants

2.1.1 Main Definitions and Notation

We begin by introducing some terminology and notation which will be used from this point on. All the matrices we deal with in this chapter, unless stated otherwise, will be square matrices.

One approach to define determinants is to use a recursive definition, that is define the determinant of an $n \times n$ matrix in terms of determinants of $n-1 \times n-1$ matrices. In turn, these determinants of $n-1 \times n-1$ matrices will be defined in terms of determinants of $n-2 \times n-2$ matrices. And so on. Of course, we do need an absolute definition for 2×2 matrices.

Definition 132 *The determinant of a 2×2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, denoted* $\det(A)$ *or $|A|$ is defined to be*

$$|A| = ad - bc \tag{2.1}$$

Example 133 *If $A = \begin{bmatrix} 1 & 4 \\ 5 & 2 \end{bmatrix}$ then $|A| = 2 - 20 = -18$.*

Recall, we saw that a 2×2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ was invertible if and only if $ad - bc \neq 0$ and $A^{-1} = \dfrac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$. In terms of determinants, this becomes: A is invertible if and only if $|A| \neq 0$ and $A^{-1} = \frac{1}{|A|}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$. We will extend the definition of determinant to larger matrices and we will see that a similar result holds. First, we need to define some intermediate terms.

Definition 134 (Minors and Cofactors) *Let $A = [a_{ij}]$ be an $n \times n$ matrix.*

1. *For each i and j, we define the* ***minor*** *of entry a_{ij}, denoted M_{ij} to be the determinant of the matrix obtained by deleting the i^{th} row and j^{th} column of A.*

2. *The* ***cofactor*** *of entry a_{ij}, denoted C_{ij} is defined to be*

$$C_{ij} = (-1)^{i+j} M_{ij}$$

Example 135 *Find M_{11} and C_{11} for $A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 7 & 1 \\ 5 & 3 & 1 \end{bmatrix}$*

- *Deleting the first row and column of A, we obtain $M_{11} = \begin{vmatrix} 7 & 1 \\ 3 & 1 \end{vmatrix} = 7 - 3 = 4$.*
- $C_{11} = (-1)^{1+1} M_{11} = 4$

Example 136 *Find M_{32} and C_{32} for A from the previous example.*

- *Deleting the third row and second column of A, we obtain $M_{32} = \begin{vmatrix} 1 & 3 \\ 1 & 1 \end{vmatrix} = -2$*
- $C_{32} = (-1)^{3+2} M_{32} = 2$

We are now ready to define determinants for square matrices of any size.

Definition 137 (Determinant) *Let $A = [a_{ij}]$ be an $n \times n$ matrix for $n \geq 2$. We define*

$$\begin{aligned} |A| &= a_{11}C_{11} + a_{12}C_{12} + ... + a_{1n}C_{1n} \\ &= \sum_{i=1}^{n} a_{1i}C_{1i} \end{aligned}$$

Remark 138 *This is called computing the determinant by cofactor expansion along the first row because we used the cofactors of the entries of the first row.*

Remark 139 *This definition gives us the definition we saw earlier if $n = 2$.*

Example 140 *Find* $|A|$ *for* $A = \begin{bmatrix} 1 & 5 & 0 \\ 2 & 4 & -1 \\ 0 & -2 & 0 \end{bmatrix}$

By definition,

$$\begin{aligned} |A| &= 1 \times (-1)^{1+1} \begin{vmatrix} 4 & -1 \\ -2 & 0 \end{vmatrix} + 5 \times (-1)^{1+2} \begin{vmatrix} 2 & 0 \\ -1 & 0 \end{vmatrix} + 0 \times (-1)^{1+3} \begin{vmatrix} 2 & 4 \\ 0 & -2 \end{vmatrix} \\ &= -2 - 5(0) + 0 \\ &= -2 \end{aligned}$$

It turns out that to compute the determinant of a matrix, we can use a cofactor expansion along any row or column. We state this as a result without proof.

Theorem 141 *The determinant of an* $n \times n$ *matrix* A *can be computed by a cofactor expansion along any row or column.*

1. *If we use an expansion along the* i^{th} *row,* $1 \leq i \leq n$*, we have*

$$\begin{aligned} |A| &= \sum_{j=1}^{n} a_{ij} C_{ij} \\ &= a_{i1}C_{i1} + a_{i2}C_{i2} + ... + a_{in}C_{in} \end{aligned}$$

2. *If we use an expansion along the* j^{th} *column,* $1 \leq j \leq n$*, we have*

$$\begin{aligned} |A| &= \sum_{i=1}^{n} a_{ij} C_{ij} \\ &= a_{1j}C_{1j} + a_{2j}C_{2j} + ... + a_{nj}C_{nj} \end{aligned}$$

The strategy is to use an expansion along a row or column with a lot of zeros.

Example 142 *Earlier, we computed* $|A|$ *for* $A = \begin{bmatrix} 1 & 5 & 0 \\ 2 & 4 & -1 \\ 0 & -2 & 0 \end{bmatrix}$ *using a cofactor expansion along the first row. Let compute* $|A|$ *using a cofactor expansion along the third column (this is a good strategy because it has two zeros).*

$$\begin{aligned} |A| &= 0 \times (-1)^{1+3} \begin{vmatrix} 2 & 4 \\ 0 & -2 \end{vmatrix} - 1 \times (-1)^{2+3} \begin{vmatrix} 1 & 5 \\ 0 & -2 \end{vmatrix} + 0 \times (-1)^{3+3} \begin{vmatrix} 1 & 5 \\ 2 & 4 \end{vmatrix} \\ &= 1(-2) \\ &= -2 \end{aligned}$$

The same result as we found before.

2.1.2 Determinants of Triangular and Diagonal Matrices

From the above theorem, one can see that computing a determinant can be quite lengthy (more on this in the section on numerical consideration). However, in the case of special matrices, this process is much easier.

Theorem 143 *If A is an $n \times n$ upper triangular, or lower triangular, or diagonal matrix, then $|A|$ is the product of the elements in its main diagonal.*
Proof. *We do a proof by induction. Note that it is enough to prove the result for triangular matrices. A diagonal matrix is also triangular (upper or lower). In fact, we do the proof for upper triangular matrices. It is identical for lower triangular.*

- *Base case. The result is true for 2×2 upper triangular matrices. Let $A = \begin{bmatrix} a & b \\ 0 & c \end{bmatrix}$. Then, by definition*

$$\begin{aligned} |A| &= ac - 0 \times d \\ &= ac \end{aligned}$$

as claimed.

- *Assume the result is true for $n-1 \times n-1$ upper triangular matrices, show it is true for $n \times n$ upper triangular matrices. Let $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ 0 & a_{22} & a_{23} & \cdots & a_{2n} \\ 0 & 0 & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \cdots & a_{nn} \end{bmatrix}$.*
Using a cofactor expansion along the first column, we see that

$$\begin{aligned} |A| &= (-1)^{1+1} a_{11} \begin{vmatrix} a_{22} & a_{23} & \cdots & a_{2n} \\ 0 & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & & \cdots & a_{nn} \end{vmatrix} \\ &= a_{11} \begin{vmatrix} a_{22} & a_{23} & \cdots & a_{2n} \\ 0 & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & \cdots & a_{nn} \end{vmatrix} \end{aligned}$$

But the determinant is that of an $n-1 \times n-1$ upper triangular matrix, hence by the induction hypothesis is equal to the product of the entries on the main diagonal, thus

$$|A| = a_{11} a_{22} \cdots a_{nn}$$

as claimed.

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Example 144 *Find* $|A|$ *for* $A = \begin{bmatrix} 1 & 3 & 5 & 7 \\ 0 & 10 & 3 & 7 \\ 0 & 0 & 2 & 8 \\ 0 & 0 & 0 & 4 \end{bmatrix}$

From the theorem, $|A| = (1)(10)(2)(4) = 80$.

Remark 145 *Note that if any of the entries on the main diagonal of a triangular or diagonal matrix is* 0*, the determinant will automatically be* 0*.*

2.1.3 Additional Related Topics

This is just given for students who want to do further reading.

Definition 146 (Adjoint) *Let* A *be an* $n \times n$ *matrix.*

1. *Remembering that* C_{ij} *denotes the cofactor of* a_{ij}*, the matrix* $\begin{bmatrix} C_{11} & C_{12} & \cdots & C_{1n} \\ C_{21} & C_{22} & \cdots & C_{2n} \\ \cdots & \vdots & \ddots & \vdots \\ C_{n1} & C_{n2} & \cdots & C_{nn} \end{bmatrix}$ *is called the* ***matrix of cofactors****. from* A*.*

2. *The transpose of the above matrix is called the* ***adjoint*** *of* A*,denoted* $adj\,(A)$*.*

Theorem 147 *If* A *is invertible, then*

$$A^{-1} = \frac{1}{|A|} adj\,(A)$$

This is a generalization of the formula we saw for the inverse of a 2×2 matrix, a few sections back.

2.1.4 Numerical Note

The idea of determinants was considered as early as 1683 by the Japanese mathematician Seki Takakazu and independently in 1693 by the German mathematician Gottfried Leibniz. In 1750, the Swiss mathematician Gabriel Cramer hinted that determinants might be useful in analytical geometry. Then, in 1812, Augustin Cauchy published a paper in which he used determinants to compute the volume of several solid polyhedra. In Cauchy's day, matrices were small and determinants easy to use. However, by today's standards, a 25×25 matrix is considered small. People handle matrices of several thousands rows and columns. Yet, it would be impossible to compute the determinant of a 25×25 matrix by cofactor expansion. It can be shown that a cofactor expansion requires $n!$ multiplications. $25! = 1.5 \times 10^{25}$. If a computer performs one trillion multiplications per second, it would take almost 500,000 years to compute such a determinant. In the next section, we will look at much faster way to compute determinants.

2.1.5 Concepts Review

- Know what the minor and cofactor of an entry of a matrix are, be able to compute them.
- Know how to compute the determinant of a 2×2 matrix.
- Know what the determinant of any square matrix is, be able to compute it by cofactor expansion along any row or column.
- Understand how to pick the best row or column to compute the determinant of a matrix.
- Know how to compute the determinant of triangular and square matrices.

2.1.6 Problems

1. Read about Takakazu, cramer, Leibniz and Cauchy.
2. On pages 94 - 96, do # 1, 2, 3, 5, 7, 29, 30.