TRANSMISSION LINE

1. Telegraph Equations

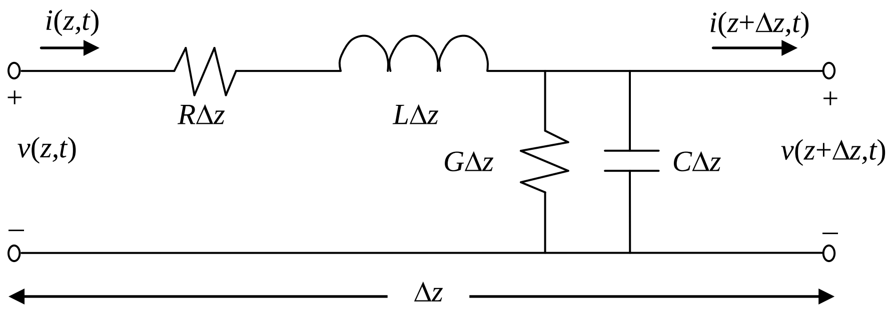


Fig.(1.1)

From the circuit of Fig.(1.1), *Kirchhoff's* voltage law can be applied to give

$$v(z,t) - R\Delta z\, i(z,t) - L\Delta z \frac{\partial i(z,t)}{\partial t} - v(z+\Delta z,t) = 0 \qquad (1a)$$

and Kirchhoff's current law leads to

$$i(z,t) - G\Delta z\, v(z+\Delta z,t) - C\Delta z \frac{\partial v(z+\Delta z,t)}{\partial t} - i(z+\Delta z,t) = 0 \qquad (1b)$$

Dividing Eq.($1a$) and ($1b$) by Δz and taking the limit as $\Delta z \to 0$ gives the following differential equations:

$$\frac{\partial v(z,t)}{\partial z} = -R\, i(z,t) - L \frac{\partial i(z,t)}{\partial t} \qquad (2a)$$

$$\frac{\partial i(z,t)}{\partial z} = -G\, v(z,t) - C \frac{\partial v(z,t)}{\partial t} \qquad (2b)$$

For the sinusoidal steady-state condition, with cosine-based phasors, Eq.(2) simplify to

$$\frac{dV(z)}{dz} = -(R + j\omega L)I(z) = -Z\, I(z) \qquad (3a)$$

$$\frac{dI(z)}{dz} = -(G + j\omega C)V(z) = -Y\, V(z) \qquad (3b)$$

Where, $Z = R + j\omega L$, impedance of the line per-unit length Ω/m
$Y = G + j\omega C$, admittance of the line per-unit length S/m

From Eq.(3)

$$\frac{d^2V(z)}{dz^2} = -Z\, \frac{dI(z)}{dz} = Z\, Y\, V(z) \qquad (4a)$$

$$\frac{d^2I(z)}{dz^2} = -Y\, \frac{dV(z)}{dz} = Z\, Y\, I(z) \qquad (4b)$$

Let $V(z) = e^{\gamma z}$, and solving for the second order homogenous differential Equation ($4a$) gives

$$V(z) = \underbrace{\vec{V}_o e^{-\gamma z}}_{incident\ \ voltage} + \underbrace{\overleftarrow{V}_o e^{\gamma z}}_{reflected\ \ current} = \vec{V}(z) + \overleftarrow{V}(z) \tag{5}$$

Where $\gamma(propagation\ constant) = \sqrt{ZY} = \sqrt{(R + j\omega L)(G + j\omega C)} = \alpha + j\beta$, $\vec{V}_o$ and $\overleftarrow{V}_o$ are the integration constants. Substituting the derivative of $V(z)$ from Eq.(5) in Eq.(3*a*) the current equation is found as

$$I(z) = \frac{\gamma}{Z}\left[\vec{V}_o e^{-\gamma z} - \overleftarrow{V}_o e^{\gamma z}\right] = \vec{I}(z) + \overleftarrow{I}(z) \tag{6}$$

Defining the *characteristic impedance*

$$Z_o = \frac{\vec{V}(z)}{\vec{I}(z)} = \frac{\vec{V}_o e^{-\gamma z}}{\frac{\gamma}{Z}\vec{V}_o e^{-\gamma z}} = \frac{Z}{\gamma} = \frac{Z}{\sqrt{ZY}} = \sqrt{\frac{Z}{Y}} = \sqrt{\frac{(R + j\omega L)}{(G + j\omega C)}} \tag{7}$$

Where α is the *attenuation constant*, and β is the *phase constant*.
For lossless transmission line (T.L), R=G=0, then $\alpha = 0$. Substitute Eq.(7) in (6) gives

$$I(z) = \frac{1}{Z_o}\left[\vec{V}_o e^{-\gamma z} - \overleftarrow{V}_o e^{\gamma z}\right] \tag{8}$$

2. Phase Velocity and Wavelength on the T.L

The time varying signal on the line is

$$v(z,t) = V(z)e^{j\omega t}$$

$$= \left[\vec{V}_o e^{-\gamma z} + \overleftarrow{V}_o e^{\gamma z}\right]e^{j\omega t} \tag{9}$$

Substituting $\gamma = \alpha + j\beta$ in Eq.(9) yield

$$v(z,t) = \underbrace{\vec{V}_o e^{-\alpha z} . e^{j(\omega t - \beta z)}}_{incident} + \underbrace{\overleftarrow{V}_o e^{\alpha z} e^{j(\omega t + \beta z)}}_{reflected} \tag{10}$$

Let the phase $\phi(t) = \omega t - \beta z$, then $\frac{\phi(t)}{dt} = \omega - \beta\frac{dz}{dt}$. Extreme value when $\frac{\phi(t)}{dt} = 0$,

$$0 = \omega - \beta\frac{dz}{dt} \quad \Rightarrow \quad \frac{dz}{dt} = \frac{\omega}{\beta}$$

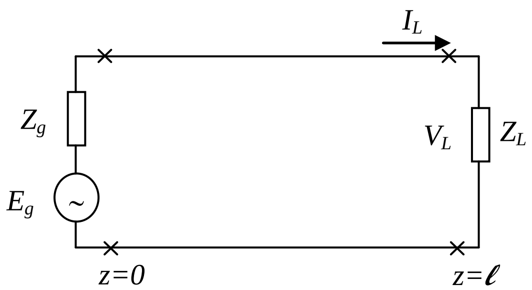
Define the *phase velocity*

$$\upsilon_p = \frac{dz}{dt} = \frac{\omega}{\beta} = f.\lambda_g \tag{11}$$

From Eq.(11), $\quad \frac{\omega}{\beta} = \frac{2\pi f}{\beta} = f.\lambda_g \quad \Rightarrow \quad \beta = \frac{2\pi}{\lambda_g}$

3. Determination of Integration Constants ($\vec{V}_o$ and $\overleftarrow{V}_o$)

In order to determine these two constants, we need to know the current and the voltage at any point on the T.L. Usually, these constants are determined either from the current and voltage at load side or at the generator side.



Let us assume that the voltage and the current at the load side (*z*=*ℓ*) V_L and I_L are known, then

From Eq.(5) and (8)

$$V_L = \vec{V}_o e^{-\gamma\ell} + \overleftarrow{V}_o e^{\gamma\ell} \quad (12)$$

$$I_L Z_o = \vec{V}_o e^{-\gamma\ell} - \overleftarrow{V}_o e^{\gamma\ell} \quad (13)$$

Adding Eq.(12) to (13), we get

$$V_L + Z_o I_L = 2\vec{V}_o e^{-\gamma\ell} \qquad \Rightarrow \qquad \vec{V}_o = \left[\frac{V_L + Z_o I_L}{2}\right] e^{\gamma\ell}$$

Subtracting Eq.(13) from (12), then

$$\overleftarrow{V}_o = \left[\frac{V_L - Z_o I_L}{2}\right] e^{-\gamma\ell}$$

Now substituting $\vec{V}_o$ and $\overleftarrow{V}_o$ in Eq.(5) yields

$$V(z) = \left[\frac{V_L + Z_o I_L}{2}\right] e^{\gamma\ell}\, e^{-\gamma z} + \left[\frac{V_L - Z_o I_L}{2}\right] e^{-\gamma\ell} e^{\gamma z}$$

$$V(z) = V_L \frac{e^{\gamma(\ell-z)} + e^{-\gamma(\ell-z)}}{2} + Z_o I_L \frac{e^{\gamma(\ell-z)} - e^{-\gamma(\ell-z)}}{2}$$

$$V(z) = V_L cosh[\gamma(\ell - z)] + Z_o I_L sinh[\gamma(\ell - z)] \quad (14)$$

Similarly, one can get for $I(z)$

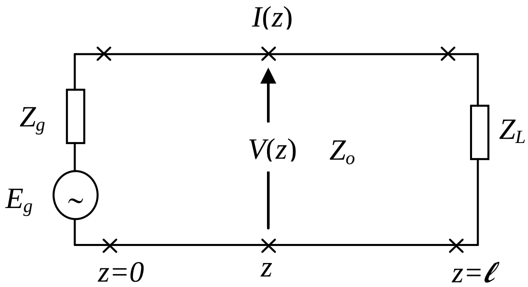
$$I(z) = I_L cosh[\gamma(\ell - z)] + \frac{V_L}{Z_o} sinh[\gamma(\ell - z)] \quad (15)$$

H.W: Try to find $\vec{V}_o$ and $\overleftarrow{V}_o$ in terms of generator side.

4. Impedance Along the T.L

The input impedance at any point along the T.L is given by;

$$Z(z) = \frac{V(z)}{I(z)} = Z_o \frac{\vec{V}_o e^{-\gamma z} + \overleftarrow{V}_o e^{\gamma z}}{\vec{V}_o e^{-\gamma z} - \overleftarrow{V}_o e^{\gamma z}}$$



or

$$Z(z) = \frac{V_L cosh[\gamma(\ell - z)] + Z_o I_L sinh[\gamma(\ell - z)]}{I_L cosh[\gamma(\ell - z)] + \frac{V_L}{Z_o} sinh[\gamma(\ell - z)]}$$

$$Z(z) = \frac{V_L + Z_o I_L tanh[\gamma(\ell - z)]}{I_L + \frac{V_L}{Z_o} tanh[\gamma(\ell - z)]}$$

dividing by I_L, we get

$$Z(z) = \frac{\frac{V_L}{I_L} + Z_o tanh[\gamma(\ell - z)]}{1 + \frac{V_L}{I_L Z_o} tanh[\gamma(\ell - z)]}$$

Since $\frac{V_L}{I_L} = Z_L$, then

$$Z(z) = \frac{Z_L + Z_o tanh[\gamma(\ell - z)]}{1 + \frac{Z_L}{Z_o} tanh[\gamma(\ell - z)]} = Z_o \frac{Z_L + Z_o tanh[\gamma(\ell - z)]}{Z_o + Z_L tanh[\gamma(\ell - z)]}$$

or

$$Z(z) = Z_o \frac{Z_L cosh[\gamma(\ell - z)] + Z_o sinh[\gamma(\ell - z)]}{Z_o cosh[\gamma(\ell - z)] + Z_L sinh[\gamma(\ell - z)]} \quad (16)$$

The impedance at $z = 0$(at the generator) is

$$Z(0) = Z_{in} = Z_o \frac{Z_L cosh\gamma\ell + Z_o sinh\gamma\ell}{Z_o cosh\gamma\ell + Z_L sinh\gamma\ell} \quad (17)$$

For lossless T.L, $\alpha = 0 \quad \Rightarrow \quad \gamma = j\beta$,
and $\cosh(j\beta\ell) = cos\beta\ell$, also $\sinh(j\beta\ell) = j\ sin\beta\ell$, therefore;

$$Z_{in} = Z_o \frac{Z_L cos\beta\ell + jZ_o sin\beta\ell}{Z_o cos\beta\ell + jZ_L sin\beta\ell} = Z_o \frac{Z_L + jZ_o tan\beta\ell}{Z_o + jZ_L tan\beta\ell} \quad (18)$$

5. Reflection Coefficient (Γ)

The amplitude of the reflected voltage wave normalized to the amplitude of the incident voltage wave is defined as the voltage reflection coefficient, Γ:

$$\Gamma_V = \frac{\overleftarrow{V}(z)}{\overrightarrow{V}(z)} = \frac{\overleftarrow{V_o} e^{\gamma z}}{\overrightarrow{V_o} e^{-\gamma z}}$$

$$= \frac{\left[\frac{V_L - Z_o I_L}{2}\right] e^{-\gamma\ell} e^{\gamma z}}{\left[\frac{V_L + Z_o I_L}{2}\right] e^{\gamma\ell} e^{-\gamma z}} = \left[\frac{V_L - Z_o I_L}{V_L + Z_o I_L}\right] e^{-2\gamma(\ell - z)}$$

Let $y = \ell - z$, then

$$\Gamma_V = \left[\frac{V_L - Z_o I_L}{V_L + Z_o I_L}\right] e^{-2\gamma y} = \left[\frac{Z_L - Z_o}{Z_L + Z_o}\right] e^{-2\gamma y}$$

At load (y = 0), $\Gamma_{Vo} = \left[\frac{Z_L - Z_o}{Z_L + Z_o}\right]$, therefore

$$\Gamma_V = \Gamma_{Vo} e^{-2\gamma y} \quad (19)$$

If $Z_L = Z_o$, results that $\Gamma_{Vo} = 0$, i.e., the load impedance is matched to the characteristic impedance.

For lossless T.L., $\Gamma_V = \Gamma_{Vo} e^{-2j\beta y}$, the reflection coefficient has constant magnitude along the line but with different phase.

Similarly the current reflection coefficient

$$\Gamma_I = \frac{\overleftarrow{I}(z)}{\overrightarrow{I}(z)} = \frac{-\frac{1}{Z_o} \overleftarrow{V_o} e^{\gamma z}}{\frac{1}{Z_o} \overrightarrow{V_o} e^{-\gamma z}} = -\frac{\overleftarrow{V_o}}{\overrightarrow{V_o}} e^{2\gamma z}$$

After substituting $\overleftarrow{V_o}$ and $\overrightarrow{V_o}$, one can get

$$\Gamma_I = -\Gamma_{Io} e^{-2\gamma y} \quad (20)$$

Therefore, $\Gamma_I = -\Gamma_V$